Vortex duality in higher dimensions

PROEFSCHRIFT

ter verkrijging van de graad
van Doctor aan de Universiteit Leiden,
op gezag van Rector Magnificus
prof. mr. P. F. van der Heijden,
volgens besluit van het College voor Promoties
te verdedigen op donderdag 1 december 2011
klokke 13.45 uur

door

Aron Jonathan Beekman

geboren te Gouda in 1979

Promotiecommissie

Promotor:	Prof. dr. J. Zaanen
Overige leden:	Prof. dr. N. Nagaosa *Universiteit van Tokyo*
	Prof. dr. A. Sudbø *Norges teknisk-naturvitenskaplige universitet Trondheim*
	Prof. dr. P.H. Kes
	Prof. dr. J.M. van Ruitenbeek
	Dr. K.E. Schalm
	Prof. dr. E.R. Eliel

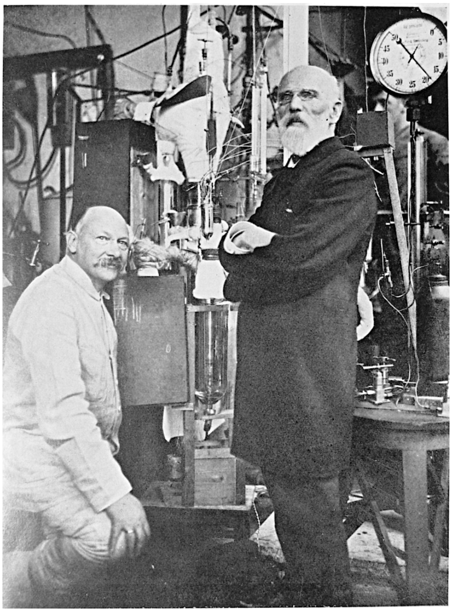
Dit werk maakt deel uit van het onderzoekprogramma van de Stichting voor Fundamenteel Onderzoek der Materie (FOM), die deel uit maakt van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO). Dit werk is tevens ondersteund vanuit een NWO Spinozapremie.

This work is part of the research programme of the Foundation for Fundamental Research on Matter (FOM), which is part of the Netherlands Organisation for Scientific Research (NWO). This work has also been supported via a NWO Spinoza grant.

Typeset in LaTeX
cover design: Rolf de Jonker / Studio Loupe
cover photo: Sander Foederer
Casimir PhD series, Delft–Leiden 2011-24
ISBN 978-90-8593-113-3

On April 8, 1911, in the physics laboratory located at 'het Steenschuur' in Leiden, Heike Kamerlingh Onnes and his coworkers Cornelis Dorsman, Gerrit Jan Flim and Gilles Holst, measured the instantaneous drop in resistivity of mercury when they cooled it below 4.2 degrees above absolute zero. They were the first people in the world to have beheld the phenomenon of superconductivity, and thereby the first macroscopic quantum fluid.

This year we have celebrated the centennial of this event. Superconductivity in its manifestation of the underlying quantum mechanical principles and its potential for world-changing applications does not cease to challenge the imagination. It is a privilege to be part of the continuing research on this fascinating topic in its place of birth.



H. Kamerlingh Onnes and J.D. van der Waals with the helium liquefactor (1911)

photo courtesy of the Leiden Institute of Physics

Contents

Chapter 1

Introduction

Condensed matter physics concerns the collective behaviour of a large number of particles that organize themselves into an ordered medium. It is the qualification 'ordered' that sets the field apart from the study of gases or simple liquids. Thus, the primary business of a condensed matter physicist is to discern when a system is ordered, and what sets apart one ordered medium from the other. One step further, she could investigate how one state can transform into another, for instance a liquid freezing into a solid or a paramagnet going over into a ferromagnet. This is the study of phase transitions, and in its modern incarnation is over 100 years old. The traditional way of thinking is always about obtaining a more ordered state (solid) from a less ordered state (liquid). This is accompanied by a *lowering* of the external or internal symmetry of the system.

It had been realized first in material science that metals start to degrade in their structural integrity but also their electronic properties by the presence of *defects*: aberrations in the regularity of the crystal lattice. If it is just a missing or superfluous particle, it is called an interstitial, and it will have limited effect on the overall properties of the material. Conversely, if the defects are *topological*, their influence has bearing throughout the whole system. Therefore those are usually confined in combinations whose topological effects cancel out each other.

The topological defects are sources of disorder. Letting more and more of these topological defects enter the system amounts to putting more disorder into it. It is tempting to continue this reasoning by stating that also the transitions into a more disordered phase are therefore caused by topological defects. That is indeed the principal perspective in this thesis. The defects are then the agents that *restore* symmetry in the system.

The alternative of focussing on disordering instead of ordering of matter is known as a *duality*. Each point of view is equally valid, and one can freely switch between the one or the other, in the ideal case via a mathematical isomorphism. Practically speaking, there are often advantages of preferring one approach over of the other, and it is therefore useful to develop both the traditional and the dual methods in order to maximize the size of the toolbox. Basically, the canonical formalism works well when the system is mostly ordered, the dual formalism when it is heavily disordered. But it is not just pragmatism that encourages the dual way of thinking; it also reveals deeper truths about the physical principles that dictate the effective collective behaviour in many-body systems.

This thesis fully embraces the dual side, and expands its applicability to higher dimensions where it was mostly restricted to the spatial plane. Let us now first get accustomed to dualities by some famous examples, in order to appreciate the problems we wish to address. Along the way we encounter many concepts that will be used copiously throughout this work.

1.1 Kramers–Wannier duality and its extensions

1.1.1 Kramers–Wannier duality

It is fitting that the first such duality was discovered in the simplest statistical physics problem: the Ising model on a square lattice. Kramers and Wannier noted that the partition function in terms of the variables $s_i \in \{-1,+1\}$ on lattice sites i, as a function of inverse temperature β, could be rewritten in terms of variables $\sigma_{\langle ij \rangle} \in \{-1,+1\}$ on the lattice links $\langle ij \rangle$ as a function of the dual inverse temperature $\tilde{\beta} \sim 1/\beta$ [1–3]. The Ising model maps to another Ising model, yet with a different coupling constant. As such, it is a mathematical identity; however, it hints to an alternative understanding of the physical principles.

This is illustrated in figure 1.1(a). The black, solid lines are the real lattice with on each lattice site arrows ("spins") that can point in two directions. Then on each link between two sites we can define a dual spin (blue) that points up if the neighbouring sites are parallel, and down if they are anti-

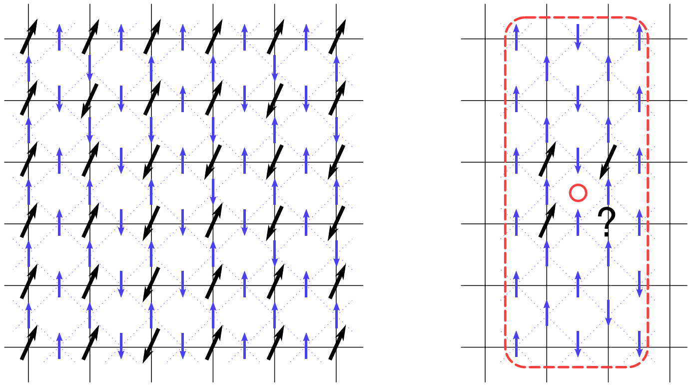


Figure 1.1: Ising spins on the square lattice (black). On each link of the lattice we can define a dual spin (blue) that points up if the two neighbouring spin are aligned and down when they are anti-aligned. The reciprocal lattice is shown in blue, dotted lines. (a) A typical configuration of spins. Note that the number of dual down spins around each plaquette is always even. (b) If we insist on having an odd number of dual down spin in the plaquette with the red circle, the original spins become frustrated. The frustration can be seen at the perimeter (dashed red), which also has an odd number of down spins.

parallel. Except for the initial condition, the dual spins contain the same amount of information as the real spins. This is the archetypical example of duality.

Now things get really interesting. While for the real spins is it perfectly fine to flip any one, possibly changing the energy but not violating any rules, notice that the number of dual spins that are pointing down around one plaquette is always even. Purely due to the definition in terms of the original spins, there is a constraint or conservation law for the dual spins. What happens if we try to break this law? This is pictured in figure 1.1(b). The red circle indicates a plaquette with only one dual down spin. If we try to recreate the original spins, starting bottom left, we see that there is no way to decide where to put the final spin around this plaquette. This plaquette is therefore said to be *frustrated*.

The frustrated plaquette is our first example of a topological defect: if one counts the number of down spins around the perimeter of our dual lattice, the number of dual down spins is also odd. The influence of the topological defect is felt all the way to the edge of the system.

The appearance of a constraint for the dual variables, and the ill-definedness of the original variables when violating this constraint is a very general principle, and one could say that this lies at the heart of all that will be discussed in this thesis.

Another recurring theme is that the dual coupling constant $\tilde{\beta}$ is inversely proportional to the original coupling constant β. This is therefore known as a strong/weak duality or S-duality. One often uses perturbation theory to be able to make calculations at all, and therefore the duality proves its worth in the high-temperature regime where $\tilde{\beta}$ is small, and can be used as the expansion parameter. This already indicates that the disordered state is actually dually ordered.

1.1.2 Ising gauge model

The basic duality of the square lattice Ising model can be extended in several ways. The energy of the Ising model above is invariant under flipping all spins at the same time—a global transformation—but local spin flips will in principle change the energy of the state. However, consider plaquette variables that count whether that plaquette has an even or odd number of dual spins down around it. Flipping all dual spins emerging from a lattice site will leave those plaquette variables invariant: the evenness does not change under such local spin flips. Instead of a global we have now a local or gauge symmetry. Any model built out of these plaquette variables will therefore have a gauge symmetry. This was first investigated by Wegner [2–4], and is called Ising gauge model or $\mathbb{Z}_2$ lattice gauge theory.

The Ising model on the square lattice is dual to another Ising model on the reciprocal square lattice. This *self-duality* is coincidental. Interestingly the Ising model on a three-dimensional cubic lattice is dual to an Ising gauge model on the reciprocal (cubic) lattice. This is known as a global/local duality: the global symmetry turns into a local symmetry for the dual variables. Also this phenomenon is a key ingredient of this thesis.

1.1.3 XY-model and the superfluid

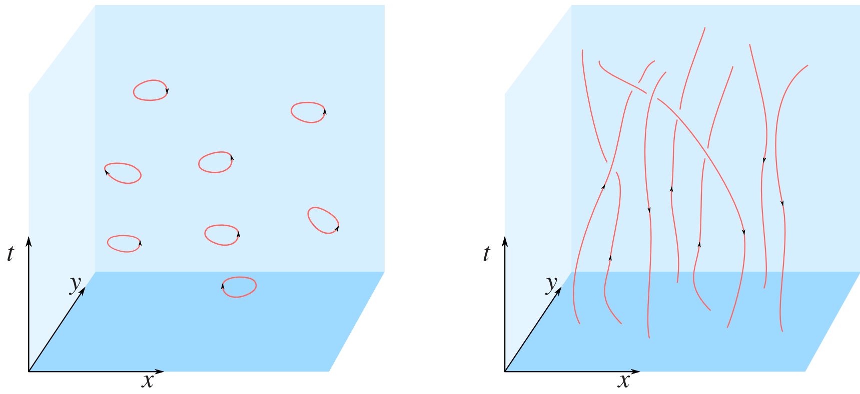
In the Ising model, the real, dual and plaquette variables take one out of two values only. This can be extended to a larger number of discrete values, but moreover to a continuous set, in particular a real or complex number. If in our picture of figure 1.1(a) the arrows on each site are of fixed length but can rotate freely in the xy-plane, then with nearest-neighbour interactions this is known as the phase-only model or XY-model. The XY-model is invariant under rotating all spins over a fixed angle α, i.e. under global $U(1)$-rotations $\mathrm{e}^{\mathrm{i}\alpha}$. An unordered XY-system has the arrows pointing in random directions whereas when their orientation is correlated over considerable length scale, it is an ordered system.

Since we are now dealing with continuous variables, we are equipped with the concept of smoothness, which shall turn out to be an essential property. Even in the ordered system, there will now be small fluctuations in the direction of the arrows around their equilibrium position, which were unavailable in the discrete systems above. Similarly, when we disturb the ordered system from the outside, this disturbance will propagate through the ordered system as the equivalent of a sound wave. This is called a Nambu–Goldstone mode, and it communicates the rigidity of the order. Goldstone modes are present in any ordered system of continuous variables—this is the famous Goldstone theorem [5–7]. Using a similar duality transformation as above, the Goldstone modes are expressed as dual gauge field, so it is a global/local duality. Here we have the natural interpretation of gauge fields are force carriers (cf. a photon), conveying the rigidity of the order parameter.

The XY-model in the continuum limit is the simplest model that describes the freely propagating zero-sound mode in a superfluid, where the arrows represent the superfluid phase variable. A superfluid in a rotating vessel will show the formation of vortices, which are in fact the topological defects. In the XY-model a vortex is a configuration where the direction of the phase changes by a multiple of 2π when traversing a closed contour. Therefore the vortices are the cause of the disordering of the phase rigidity. Surely when the external angular momentum gets too large, the superfluid will be destroyed entirely by the induced vortices.

1.1.4 Vortex unbinding transitions

But in the duality viewpoint, also thermal (or quantum) fluctuations cause spontaneous formations of small vortex loops. These loops grow with rising temperature, and then the thermal phase transition is also understood as



(a) small vortex loops (b) vortex blowout

Figure 1.2: The phase transition in terms of vortex world lines. (a) In the ordered phase, the dual coupling constant (the vortex line tension) is large, such that it is very costly to form vortex lines. In spacetime they only appear as small loops of creation and annihilation of vortex–anti-vortex pairs. (b) Increasing disorder amounts to lowering the dual coupling constant, so that the vortex loops grow. Across the phase transition the loops have grown to the system size, and the proliferate throughout the whole system. This picture should always be kept in mind when reading this thesis.

the demise of order due to vortices.

This is best understood pictorially. At low temperatures, the formation of vortex pairs will be heavily suppressed, and only small spacetime loops of vortex–anti-vortex pairs will appear (Fig. 1.2(a)). But as temperature rises, it becomes entropically favourable to let the vortex lines grow—this is the dual equivalent of the increasing population of excited states with phase orientations different than the purely ordered ground state. At the critical temperature, these loops grow to be of the system size, and energetically the vortex lines can now permeate the system freely (Fig. 1.2(b)). The phase (the arrows) is completely disordered. This is referred to as the "vortex blowout" or the "tangle of vortex world lines" and the phase transition is the "vortex unbinding transition".

In principle, the discrete model like the Ising models also undergo a defect-unbinding transition, but the effect is more striking in the continuous models: in 1966 Mermin and Wagner showed that a two-dimensional magnetic system will always disorder due to thermal fluctuations; this actually holds for any two-dimensional system, and is known as the Mermin–Wagner–Hohenberg–Coleman theorem [8–10]. Therefore it came as a surprise when Kosterlitz and Thouless, and independently Berenzinskii, showed that there is a vortex unbinding transition in the two-dimensional XY-model [11, 12]. This is commonly explained as: “this phase transition is not a order–disorder phase transition”. I will have some comments on this issue in the conclusions, chapter 7. This theme was expanded to external (spatial) symmetries by Nelson, Halperin and Young, which had the most impact for classical liquid crystals [13, 14]. In this context one speaks of defect-unbinding transitions or defect-mediated melting.

Vortices are pointlike in two spatial dimensions, and the mnemonic for the BKT transition is, also in 2+1 dimensions, the picture of Fig. 1.2. But a vortex is a line in three spatial dimensions. Still the phase transition cannot be anything different than the disordering of the phase variable. The question arises if one can generalize the vortex blowout when the vortices are not points but extended objects. We will show in chapter 3 that that is indeed the case.

1.1.5 Phase transitions with gauge fields

If one were to promote the global $U(1)$-symmetry of the superfluid to a local or gauge symmetry, this necessitates the introduction of a vector-valued gauge field. This is precisely the situation in the superconductor, where the massless photon field A_μ, a vector field with gauge symmetry, couples to the superconducting phase, the Goldstone modes. The gauge field then undergoes the famous Anderson–Higgs mechanism [15], and becomes massive. As a result, the photon field is expelled from the superconductor, and there are only massive, gapped excitations in the medium. Also the interactions between vortices in the superconductor become short-ranged, which shows in the correlation functions of the *dual* variables. The simplest model that features the Higgs mechanism is the Abelian-Higgs model, and in 2+1 dimensions this is precisely how the vortex unbinding transition works. Therefore, vortex duality often goes by the name of Abelian-Higgs duality, and the disordered XY-phase is in this context a “dual superconductor”.

Now a field with a local symmetry cannot undergo a phase transition (spontaneous symmetry breaking) by itself, this is Elitzur’s theorem [16].

Therefore it seems natural to argue that, in the superconductor, first the 'superfluid' order is established, and secondarily the gauge field follows the symmetry breaking by coupling to the Goldstone mode. That is indeed the point of view we will take in this thesis, and will even prove to be more than just an equivalent description when identifying the massive modes in the 3+1 dimensional disordered superfluid and superconductor (chapters 3 and 5).

1.1.6 Going quantum

In recent years there has been increasing interest in phase transitions due to the disordering effect of quantum fluctuations instead of thermal fluctuations. Such phenomena are called *quantum phase transitions* [17].

It has long been noted (e.g. by Feynman [18]) that the quantum mechanical weight factors in the path integral are just like Boltzmann factors if one transforms to imaginary time $t \to \mathrm{i}\tau$. As such, as a dynamical quantum field theory in D dimensions is easily mapped to a statistical mechanics problem in $D+1$ dimensions, where the role of time is played by temperature. This correspondence was originally used to carry over knowledge from thermal physics to quantum many-body systems; for instance the textbook by Mahan carries out many calculations at a finite temperature, to let temperature go to zero at the very end [19].

In quantum phase transitions this is taken one step further. It is not just equilibrium physics, but also phase transitions that are closely mimicked. One now has a coupling constant that represents the strength of quantum fluctuations, and which is therefore the analogue of the temperature. For instance, in high-temperature superconductors it is the number of free charge carriers that plays this role (see §5.1.2). Despite the numerous similarities, quantum phase transitions are more intricate and eventually richer than thermal ones.

This is most prominently seen by the phenomenon of spontaneous symmetry breaking. In second-order phase transitions, the system spontaneously chooses one of many ground states, for instance one particular direction of the $U(1)$-spins. It will cost a lot of energy to change this order: it is rigid. In classical, thermal systems, only one direction can be chosen. But in quantum systems, any superposition of ground states is just as valid. Therefore, the quantum system allows for much more interesting ordering patterns. Also the excitation spectrum is affected in a similar way.

In much of what follows, the quantum nature of the phase transition is not really emphasized. One needs to keep in mind though, that the system under investigation are inherently quantum mechanical in nature, and they are dominated by the Goldstone modes arising from quantum rigidity. Only in chapter 6 will be make a sharp distinction between classical and quantum systems, and discussion on the classicalness of quantum system takes place in §7.2.4.

1.1.7 Other dualities

Up to know we have only discussed the simplest dualities: strong/weak and local/global dualities in the Ising model and in $U(1)$-symmetry, which is the simplest continuous symmetry, and is Abelian, i.e. two consecutive symmetry transformations commute.

Higher symmetry groups, especially non-Abelian groups such as $SU(2)$ for spins, are much more complicated; in particular, the "braiding" of vortex (world) lines follows the symmetry structure, and may also be non-Abelian. While this opens up many interesting avenues such as in the fractional quantum Hall effect and topological quantum computation, it leads to ambiguities in defining the tangle of vortex world lines as the disordered state. There has be some progress on the mathematical side using quantum groups or Hopf algebras [20–26].

Dualities are prevalent in string theory, in fact they are one the appealing mathematical features of that framework. In this context, the strong/weak duality is called S-duality. In several instances it relates one string theory to another. The underlying principle is the same: local variables one side turn into extended or topological objects on the other side, which unbind as the dual coupling constant grows smaller.

Almost all of what follows focussed on the Abelian $U(1)$-symmetry. Only in chapter 6 we will passingly address the space groups of general relativity and of elasticity. Trouble is avoided by focussing on the translations subgroup, which is Abelian.

1.2 The road to higher-dimensional vortex duality

The application of the Abelian-Higgs duality to many-body physics had been identified and studied for over three decades [27–36]. Even though it is still unfamiliar to many researchers in the field, once the basic concept has been grasped, the framework is quite simple and rather elegant. One reason why it remains to reside in relative obscurity may be that it has not really led to new predictions, but had been confined to placing known results in a different light.

Furthermore, vortex duality has been mostly restricted to 2+1 dimensions. The reason, which we shall discuss extensively in §2.2, is that in that case the vortices act just as point particles do: in spacetime they trace out world lines, and we capture those in a regular quantum field theory. In higher dimensions, the vortices becomes extended objects like lines or surfaces. As long as they are distant from each other (strong coupling limit of the vortices), the duality works fine: dual gauge fields mediate interactions between individual vortex sources. The dual gauge field is just the Hodge dual of the Goldstone scalar field, i.e. a free $d-2$-form field, and the dynamics of such a free tensor field is well known (see e.g. [37]).

However trying to effect the phase transition is really difficult. One wishes to form a condensate of the extended vortex world sheets, in which their number is no longer conserved. In other words: we are looking for a quantum field theory of extended objects. This is the subject of string field theory, and its progress has been severely limited [38, 39]. This was recognized for instance in Ref. [40, §2.5], and therefore not pursued any longer.

It is amusing to trace back how this work was initiated originally. The correspondence between spacetime deformations of general relativity (GR) and elasticity in crystals has been noted by many authors. In recent years, the mathematical physicist Hagen Kleinert has explored this relation in depth by imagining a “world crystal” deformed by topological defects [41, 42]. Then the defects are like sources of curvature and the stress tensor corresponds to the Einstein tensor. He also recognized that the dynamics is slightly off, leading to wrong correlation functions, and tried to solve this with a “floppy world crystal”, which is in a sense “looser” than an ordinary crystal. The deeper reason is that even in the continuum limit the crystal remembers that both translational and rotational symmetry are broken, while GR is practically translationally invariant (see §6.2).

A different proposal was put forward by Kleinert and Zaanen [43] that GR does not correspond to a crystal, but to a quantum liquid crystal. In such a material, part of the spatial symmetry is restored by reviving translational invariance. The route to this symmetry restoration is precisely via the unbinding of topological defects, in this case the crystals dislocations. While the claim was made that this should hold for any dimension, the calculation was done in 2+1 dimensions only. However, gravity in 2+1 dimensions is simple, or boring, in the sense that there are no propagating modes—no gravitons. The real magic happens in four spacetime dimensions, where GR predicts two graviton polarizations as massless spin-2 modes. Gravitons have not been detected directly, and a huge effort is currently invested to find them in the form of gravitational waves [44].

Thus, I set out to identify the hydrodynamic modes of quantum liquid crystals in 3+1 dimensions that should correspond to gravitons, building upon the work done by Cvetkovic and Zaanen in 2+1 dimensions [31, 40, 45]. Many parts are readily generalized to higher dimensions, but soon we bumped into the obstacle mentioned above: the unbinding transition of extended topological defects in higher dimensions. Therefore it was necessary to take a few steps back, to really comprehend where the difficulties in the vortex duality lie.

It was fortunate that at about that time, Marcel Franz just published a work on this topic [46], continuing an idea by Rey [47] into the realm of condensed matter physics. Some research had in fact been done, starting with Marshall & Ramond in the context of string theory [48]. These attempts take the dual gauge field as the central object and start from there. Then it is logical to extrapolate the Anderson–Higgs mechanism from vector fields to tensor fields and suggest that the vortex condensate will turn the higher-form gauge field massive. Another approach was taken in Ref. [49].

However, we soon noted that there was a flaw in this argument, which leads to an overcounting in the number of degrees of freedom. In condensed matter physics we often have the advantage that the systems under consideration are accessible in the laboratory, in computer simulations, and by several theoretical approximations. As such, we knew that the vortex duality in the continuum XY-model should eventually reproduce the results of a firmly established lattice model, namely the Bose-Hubbard model. This model has been realized almost perfectly in cold atom experiments [50].

Our guiding principle was therefore to obtain the characteristics of the Mott insulating state (the strong-coupling limit of the Bose-Hubbard model) from the vortex condensate, in particular a doublet of degenerate gapped modes in any dimension. In other words: we needed to generalize those properties of the vortex duality from 2+1 to higher dimensions that carry the information of these massive propagating modes. Naively Higgsing the dual tensor gauge field will not do this for you. We were finally able to perceive that one should focus on the conserved currents rather than on the dual gauge fields, and this enabled a comprehensive generalization of the vortex duality which should hold for any order–disorder transition in condensed-matter systems in any dimension higher than two (chapter 3).

As we struggled through unexplored territory, it became clear that the vortex lines as spacetime world sheets interacting via dual gauge fields contain a huge amount of information that can be extracted directly in the dual language. In condensed matter physics, most work on vortices is related to laboratory-based setups in superconductors and superfluids, and the mathematical niceties of extended defects that play a large role in for instance cosmology and string theory are glossed over or not even acknowledged. Conversely the fact that a vast body of knowledge on vortex lines has been collected does not reflect back on the high-energy community, which is demonstrated by the unwillingness to admit what are called Nielsen–Olesen strings are just relativistic Abrikosov lines, and what is called the Abelian-Higgs model is just relativistic Ginzburg–Landau theory.

In this light, even on the weakly-coupled side of the phase transition, the electrodynamics of Abriksov vortices turns out to be completely contained in the dual, relativistic description of the vortex world sheet. By incorporating the time direction on even footing, the well-known magnetic equations are directly generalized into similar equations for the electric field. It turns out that all basic effects of vortex electrodynamics are captured in a single equation, which is the subject of chapter 4.

The most interesting aspect of the duality is that it is truly dual: the vortex condensate supports vortices of its own, and when these condense we are back to the original weak-coupling phase (see §§2.4.6, 3.4.5). This is not just an enjoyable gimmick, but moreover a true physical prediction. Already present in neutral 2+1 dimensional systems, it is most striking in charged 3+1 dimensional systems. Where the defects in superconductors are Abrikosov vortex lines of magnetic flux, the duality suggests vortex lines of electric current in Bose-Mott insulators. This unexpected result may be directly accessible for experimentalists to find in underdoped cuprate superconductors, and will be investigated in chapter 5.

Now that the vortex duality has be generalized to higher dimensions for the simple $U(1)$-symmetry superfluids and superconductors, we can finally direct it to the original problem of gravitons in quantum liquid crystals. Unfortunately the full calculation is not yet completed to include in this thesis. However, the wisdom acquired in chapter 3 demotes that calculation in terms of gauge fields to be of secondary importance, at least in the relativistic limit. The conserved currents (the stress tensor c.q. Einstein tensor) dictate the physical content of the model, and symmetry considerations do the rest. In fact, the local conservation law of the currents and the emergence of a conserved quantity in the vortex condensate, i.e. the density of the vortex liquid, are in harmony and closely connected. This general principle follows from the duality construction, and is established in condensed matter physics under the name of emergent gauge invariance. In the final chapter 6 we will show that the two gauge principles are indeed opposite sides of the same coin. Then we come full circle by illustrating these emergence phenomena in quantum liquid crystals, and show that a quantum nematic liquid crystal has to correspond to the linearized approximation of gravity, containing the two graviton modes.

We shall start off with one chapter of preliminary material (ch. 2) that collects known results on which the rest of the work is built. In the concluding part (ch. 7) we summarize all obtained results, try to contextualize their impact on condensed matter physics and beyond, and present open questions and new waypoints as directions of research.

1.3 Conventions

vortex duality I shall use the term “vortex–boson duality” throughout this thesis, or “vortex duality” *tout court*. This exactly describes what is happening, and is completely unambiguous. Alternative names are “XY-duality” or “Abelian-Higgs duality”. The latter is only applicable in 2+1 dimensions, but quite common in the literature. Furthermore here we dualize the Goldstone boson of the Abelian-Higgs model, whereas in high-energy physics often the

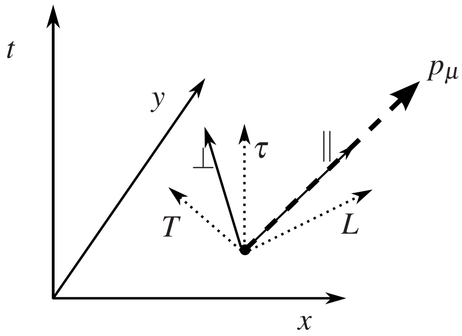


Figure 1.3: We often use two coordinate systems related to the momentum p_μ of the gauge particle. In the (τ, L, T)-system (dotted lines), the temporal direction is preserved, and the spatial ones are separated in longitudinal and transversal. This system is useful in the Coulomb gauge and when Lorentz invariance is broken. In a relativistic context, more appropriate is the $(\|, \perp, T)$-system (solid lines), where the τ and L-directions are rotated so that one is parallel to the spacetime momentum p_μ. This direction $\|$ is also called longitudinal. The spatial-transversal directions are the same as in the previous system. In higher dimensions $D+1$, there are simply more spatial-transversal directions $T_1, \ldots, T_{D-1}$.

gauge field is dualized.

metric In relativistic expressions we use the "spacelike convention" for the Minkowski metric: $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. The reason is that the spatial parts will carry the same sign as the quantities in common static, non-relativistic expressions, such as the Hamiltonian. We will often work in imaginary time $t \to i\tau$, with Euclidean metric $\delta_{\mu\nu} = \text{diag}(1,1,1,1)$. Then the integrand in the path integral reads $e^{iS/\hbar} \to e^{-S_E/\hbar}$ and looks like a Boltzmann factor. For the momentum $i\partial_\mu \to p_\mu$ we use $p_\mu = (p_\tau, \mathbf{q}) = (\frac{1}{c}\omega, \mathbf{q})$. In imaginary time the frequency here is strictly speaking a Matsubara frequency ω_n, but unless there is room for confusion, we suppress the label n.

Fourier components It is often useful to use coordinate systems related to Fourier components, as shown in figure 1.3.

units Wherever dimensionful quantities are present we express them in SI-units, for which the Ampère–Maxwell law reads,

$$\nabla \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{E} = \mu_0 \mathbf{J}. \tag{1.1}$$

This is to be compared to this relation in the quite common Gaussian cgs-units,

$$\nabla \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{E} = \frac{4\pi}{c} \mathbf{J}. \tag{1.2}$$

The reason for this choice is that in relation to experiments it is easier to refer to Ampères than to statcoulombs per second. Additionally it will turn out to be quite useful to keep around the magnetic constant μ_0, as it signals contributions from the Maxwell electromagnetic field as opposed to electric current due to moving charges.

current There will repeatedly appear two kinds of sources or currents in this thesis: the electromagnetic current (density) and the vortex current. Since they do both act as current/sources in the equations, both are represented by some form of the conventional letter J. For clarity, the vortex current will always carry a superscript label $^{\mathrm{V}}$ to distinguish is from the material current in superconductors and Mott insulators. Vortex currents in the superfluid/superconductor are denoted by the Roman symbol J^{V}, and in the Mott insulator by the script symbol $\mathscr{J}^{\mathrm{V}}$.

spacetime dimensions A capital letter "D" will be used when referring to exclusively spatial dimensions, and a small letter "d" when referring to spacetime dimensions. Thus a 2D particle traces out a world line in 2+1d spacetime.

Chapter 2

Preliminary material

Here we present some material that is not at all new, but on which later parts of this work are based. We only include discussion as far as is needed to understand the following chapters of this work.

2.1 The Ginzburg–Landau model

Here we shall very shortly recap the overly familiar Ginzburg–Landau model of superconductivity, because all of the following work will use the same order parameter language. As such it is good to set the stage such that one can always compare with well-established results, see for instance Refs. [28, 51].

2.1.1 Superfluid

In 1937 Lev Landau proposed a phenomenological field-theoretical model that was capable of capturing the essential features of continuous or second order phase transitions. It centred around the concept of an *order parameter* $\Psi(x)$, which is a function on every point in space, i.e. a field. It is capable of distinguishing between ordered and disordered phases: in the disordered phase, its average or expectation value is zero $\langle\Psi\rangle = 0$, while in the ordered phase it is non-zero $\langle\Psi\rangle = \Psi_0 \neq 0$. Landau established the simplest form that can show this behaviour,

$$E = \int \mathrm{d}^3x \; \frac{1}{2}|\nabla\Psi|^2 + \frac{1}{2}\alpha|\Psi|^2 + \frac{1}{4}\beta|\Psi|^4. \tag{2.1}$$

Here Ψ is a complex scalar field. The first term represents fluctuations in the order parameter, and can therefore be regarded as the kinetic energy. The second term is as a mass for the order parameter, and the third causes the energy to always be bounded from below. When $\alpha > 0$ the potential energy is minimized by $|\Psi| = 0$ and we are in the disordered phase. But when $\alpha < 0$, the potential energy has minima at $|\Psi| = \pm\sqrt{|\alpha|/\beta}$. This is sometimes called the 'Mexican hat potential'.

Because $\Psi = |\Psi| e^{i\varphi}$, the phase can still freely fluctuate, and in the so-called London limit where the amplitude is fixed everywhere $|\Psi|(x) = \Psi_0$, the energy reduces to,

$$E = \int d^3x \, \frac{1}{2g} (\nabla\varphi)^2, \tag{2.2}$$

modulo constant terms, and g poses as the coupling constant. This very simple model actually describes the dynamics of a superfluid, with the massless zero sound mode φ and massive density fluctuations $|\Psi|$.

The parameter α is usually taken as a function of temperature, changing sign at the critical temperature T_c. This model then also contains the scaling laws at the critical point up to the mean field level, and as such partly explains universality, the phenomenon that microscopic details are often unimportant in capturing the collective behaviour of many-body systems.

2.1.2 Superconductor

It was not until 1950 that this powerful concept was extended to charged superfluids, i.e. superconductors with the help of Vitaly Ginzburg. This was done by minimal coupling to the electromagnetic gauge potential,

$$E = \int d^3x \, \frac{\hbar^2}{2m^*} |(\nabla - i\frac{e^*}{\hbar}\mathbf{A})\Psi|^2 + \alpha|\Psi|^2 + \frac{1}{4}\beta|\Psi|^4 + \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2. \tag{2.3}$$

Here m^* and e^* are the mass and the electric charge of the charge carriers (Cooper pairs as we know now). From this energy functional, we can derive the Ginzburg–Landau equations of motion,

$$-\frac{\hbar^2}{2m^*}(\nabla - \frac{e^*}{\hbar}\mathbf{A})^2\Psi + \alpha\Psi + \beta|\Psi^2|\Psi = 0 \tag{2.4}$$

$$\frac{1}{\mu_0}\nabla \times \nabla \times \mathbf{A} - \frac{\hbar e^*}{m^*}|\Psi|^2(\nabla\varphi - \frac{e^*}{\hbar}\mathbf{A}) = 0 \tag{2.5}$$

From the first equation, when there are no external electromagnetic fields present, one can derive the *coherence length* $\xi = \frac{\hbar^2}{m^*|\alpha|}$ as the typical length scale over which the value of $|\Psi|(x)$ still fluctuates.

By action with the curl operator $\nabla\times$ on the second equation, using the definition of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and the Maxwell equation $\nabla \cdot \mathbf{B} = 0$, one finds,

$$\lambda^2 \nabla^2 \mathbf{B} - \mathbf{B} = -\frac{1}{2\pi} \Phi_0 (\nabla \times \nabla)\varphi. \tag{2.6}$$

Here we defined the *London penetration depth* $\lambda = \sqrt{\frac{m^*}{\mu_0 |\Psi|^2 e^{*2}}}$ and the *flux quantum* $\Phi_0 = h/e^*$. When the phase field φ is smooth the right-hand side vanishes, and this equation then tells us that the magnetic field is expelled from the superconductor, as it falls off exponentially over length scale λ. This is called the Meissner effect. We also identify,

$$\mathbf{J}_\mathrm{s} = -\frac{\delta E}{\delta \mathbf{A}} = \frac{\hbar e^*}{m^*} |\Psi|^2 (\nabla \varphi - \frac{e^*}{\hbar} \mathbf{A}), \tag{2.7}$$

as the supercurrent. Then Eq. (2.5) can also be written as,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_\mathrm{s}, \tag{2.8}$$

which is the non-dynamic part of the Ampère–Maxwell law. Furthermore, acting with the curl operator on Eq. (2.7), we find the London equation,

$$\nabla \times \mathbf{J}_\mathrm{s} = -\frac{1}{\mu_0 \lambda^2} \mathbf{B}. \tag{2.9}$$

where we again have used $(\nabla \times \nabla)\varphi = 0$ for a smooth phase field.

It was Alexei Abrikosov's great insight [52] that when $\lambda > \xi/\sqrt{2}$, it is energetically more favourable to let the magnetic field penetrate through vortex lines than to expel it altogether. Such a material is called a type-II superconductor. We will see in the next section that in the presence of vortices, φ becomes multivalued, and then we should identify $(\nabla \times \nabla)\varphi = 2\pi \delta^{(2)}(\mathbf{x}) N$, a 2-dimensional delta function in the plane orthogonal to the vortex line times the winding number N (see also §2.2.3). Eq. (2.6) then shows that the vortex line *is* magnetic field, that falls off exponentially away from the centre.

We can take a line integral of (2.5) deep within the superconductor where $\mathbf{B} = 0$ over a closed contour $\mathscr{C}$ around a vortex line. We find using Stokes' theorem,

$$\int_{\mathscr{S}} \mathrm{d}\mathbf{S} \cdot \mathbf{B} = \oint_{\mathscr{C}} \mathrm{d}\mathbf{x} \cdot \mathbf{A} = \frac{1}{2\pi} \Phi_0 \oint \mathrm{d}\mathbf{x} \cdot \nabla \varphi = \Phi_0 \int_{\mathscr{S}} \mathrm{d}\mathbf{S}\ \delta^{(2)}(\mathbf{x}) = \Phi_0 N. \tag{2.10}$$

Here $\mathscr{S}$ is the area enclosed by the contour $\mathscr{C}$. Thus we see that the magnetic flux through $\mathscr{S}$ and therefore through the vortex line is quantized in units of Φ_0.

The electrodynamics of Abrikosov vortices is derived from a relativistic field theory in chapter 4.

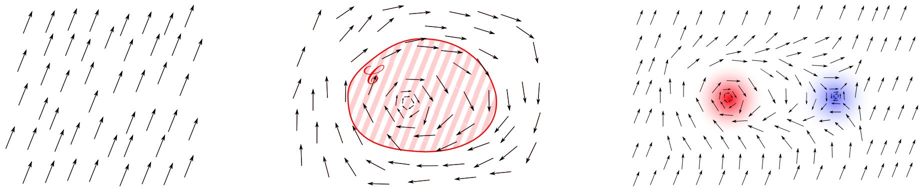
2.2 Topological defects

Once one finds oneself in an ordered state, a natural question is how it can be made disordered. Disorder is caused by defects, a simple example of which would be an interstitial atom or ion in an otherwise perfectly regular crystal lattice. Such defects cost energy to make, but usually only a fixed amount independent of the system size. As such their disordering capabilities are also not that great. It turns out that most forms of disorder are due to *topological* defects, the energy of which increases with the system size. They are thus energetically very expensive, and will in strongly ordered systems only appear in confined combinations, often pairs, which are said to be *topologically neutral*. Increasing disorder amounts to deconfining such pairs (see §1.1.4).

To understand what topological defects are and how they are classified for a specific ordered medium, one needs the mathematical machinery of homotopy theory. It explores the concept of continuity, which turns out to be *the* property of relevance in describing ordered states and the topological defects they can support. We shall not delve deeply into these matters; a good introduction is found in the review by David Mermin [53]. Here we will quote some of the results as needed for the Abelian $U(1)$-symmetry we are exclusively interested in.

2.2.1 Order parameter space

As explained above in §2.1.1, an order parameter is a continuous function on every point in space. If there are long-range correlations between the values of this function, the state is said to be ordered. The domain of the function is called "order parameter space" $\mathscr{M}$, and it can be a number, vector or any continuous manifold. We are interested in superfluids and superconductors, with order parameter a complex scalar field $\Psi = |\Psi| e^{i\varphi}$. In the completely ordered state the amplitude obtains a so-called vacuum expectation value



(a) Phase ordered (b) Single vortex (c) Vortex pair

Figure 2.1: Configurations of the phase field in the plane. (a) A trivial state with the phase perfectly ordered. (b) Configuration with a $N = 1$ vortex present. Taking the line integral around the contour $\mathscr{C}$ will give 2π. The contour and therefore the hatched area are arbitrary along as they comprise the vortex core. (c) A vortex–anti-vortex pair. Far away from these vortices the phase is ordered, and therefore this configuration is topological neutral.

(VEV) that is non-zero and constant throughout the medium. The phase of Ψ has long-range correlations. Small fluctuations around this VEV cost some energy but vanishingly little as the fluctuations die out. These fluctuations are actually the Goldstone modes, and it is easy to see that they can only arise for continuous order parameters, as there is no such thing as a small fluctuation in a discrete set. The Goldstone modes communicate the rigidity of the order parameter.

Let us first take the example of the $U(1)$ order parameter to illustrate the principles. When the amplitude $|\Psi|$ has obtained an expectation value, then only the phase φ is left, which can be pictured as an arrow on every point in space. If the system is spatially 2-dimensional, the order parameter space can be conveniently drawn just on real space. Consider the configurations in figure 2.1. Without a defect present, the phase is perfectly ordered, barring small fluctuations. When however the phase around a closed contour makes a 2π rotation, there must be a singular point where the phase is not well defined. This point is the topological defect, called a vortex for a $U(1)$-field. Wherever we draw this contour, the phase change is always 2π, which is the reason for the denomination 'topological'. We also see that a configuration of a vortex and an anti-vortex together is topologically neutral.

2.2.2 Homotopy groups

In the general case, due to thermal or quantum fluctuations, the system is free to explore part of configuration space by small perturbations around the present, ordered, state. As such we can define configurations to be equivalent if they differ by *continuous deformations* only. All of configuration space is then divided up in equivalence classes, and one class cannot be transformed into another continuously. There is one trivial class, and all the others are said to contain topological defects. It turns out that the equivalence classes are classified by the *homotopy groups* of the order parameter space. Mathematically, the nth homotopy group $\pi_n(\mathscr{M})$ has as elements all the different ways in which an n-sphere S_n can be mapped onto the space $\mathscr{M}$. For instance the first homotopy group (or fundamental group) $\pi_1(\mathscr{M})$ classifies how 'lassos' can or cannot be contracted into a point.

From the drawings in figure 2.1, we see that such lassos characterize *point* defects in a 2-dimensional plane. But in 3 dimensions, we would be able to pull the lasso 'over' the singular point. If we had a singular line, the lasso cannot be contracted. For this reason, the n-th homotopy group classifies $D-n-1$-dimensional defects in D-dimensional space. Thus π_1 classifies point defects in 2D and line defects in 3D; and π_2 classifies point defects in 3D. Now it is a result of homotopy theory that $\pi_n\big(U(1)\big)$ is isomorphic to the trivial group except for $n=1$, where it is the set of integers representing the winding numbers. Therefore the only topological defects possible are point defects in 2D and line defects in 3D, both characterized by the winding number N.

2.2.3 Multivalued fields

Almost all of the properties of vortices (or topological defects in general) can be ascribed to the singular point or line in the vortex core. The singularity is by definition not well-behaved analytically. Yet it turns out to be very fruitful to try and apply field-theoretical techniques as much as we can. In fact this is the central topic of Kleinert's textbooks [28, 41, 42]. For us it suffices to establish the following identity. The phase winds in units of 2π around the vortex core, by traversing contour $\mathscr{C}$. Thus the change of of the phase adds up to $2\pi N$,

$$\oint_{\mathscr{C}} \mathrm{d}\varphi = \oint_{\mathscr{C}} \mathrm{d}x^m\ \partial_m \varphi = 2\pi N. \tag{2.11}$$

Let $\mathscr{S}$ be the area enclosed by $\mathscr{C}$. In 3D it has a normal k that is parallel to the vortex line. Then we formally apply Stokes' theorem,

$$2\pi N = \oint_{\mathscr{C}} \mathrm{d}x^m \ (\partial_m \varphi) = \int_{\mathscr{S}} \mathrm{d}S^k \ \epsilon_{knm} \partial_n (\partial_m \varphi). \tag{2.12}$$

Thus, if there is a vortex present $N \neq 0$ the left-hand side is not zero, and we must conclude that the derivatives of the singular field φ do not commute. Therefore we are led to identify,

$$\epsilon_{knm} \partial_n \partial_m \varphi(x) = 2\pi N \delta_k^{(2)}(x). \tag{2.13}$$

Here $\delta_k^{(2)}(x)$ is a 2-dimensional delta function in the plane orthogonal to k centred around the vortex core. Since away from the core the phase field is smooth, the non-vanishing contribution is indeed purely attributable to the singular point itself. In the sequel, we shall often split the phase field in a smooth and a multivalued part,

$$\varphi = \varphi_{\text{smooth}} + \varphi_{\text{MV}}, \tag{2.14}$$

where $\epsilon_{knm} \partial_n \partial_m \varphi_{\text{smooth}}(x) = 0 \ \forall x$, whereas the multivalued part satisfies the relation above. Even though the derivatives of a multivalued field do not commute, it does satisfy the integrability condition, [28, 42],

$$\partial_k (\epsilon_{knm} \partial_n \partial_m \varphi) = 0. \tag{2.15}$$

Regarded as a physical field, we define,

$$J_k^{\text{V}} = \epsilon_{knm} \partial_n \partial_m \varphi = 2\pi N \delta_k^{(2)}(x), \tag{2.16}$$

as the *vortex current*. It is conserved $\partial_k J_k^{\text{V}} = 0$, because of the integrability condition above. These vortex currents are the central topic of this thesis.

2.2.4 Vortex world lines and world sheets

We have seen that for $U(1)$-fields there are pointlike vortices in 2-dimensional and linelike vortices in 3-dimensional space. Now we regard these objects as physical entities as moving in spacetime. The 2D vortex point (vortex pancake in superconductivity parlance) then traces out a world line in spacetime, just as any particle would. But the 3D vortex line traces out a world sheet. This is pictured in figure 2.2. In 2+1d the direction orthogonal to

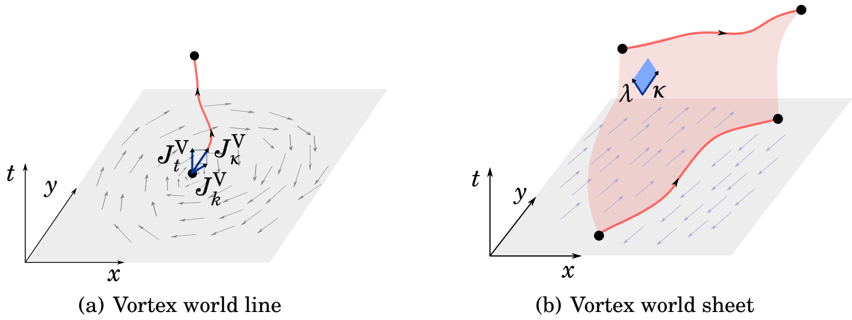


Figure 2.2: Vortices in 2+1d and 3+1d (a) A point vortex will trace out a world line. The line element J^V_κ can be decomposed in a temporal density part J^V_t and a spatial current part J^V_k. (b) In 3D we have a vortex line, here in the xy-plane, since the third spatial dimension cannot be drawn. The world sheet is built up out of surface elements $J^V_{\kappa\lambda}$. The temporal components J^V_{tl} represents the density of vorticity of the line along l, and the spatial components J^V_{kl} are the flow in direction k of the line along l.

the plane is always the time direction, but in a relativistic treatment we consider the vortex current $J^V_\kappa = \epsilon_{\kappa\nu\mu}\partial_\nu\partial_\mu\varphi_{\text{MV}}$ where the indices take values in (t,x,y). $J^V_\kappa(x)$ is just the line element of the vortex world line at x. The temporal component J^V_t is the density of vorticity defined in Eq. (2.16). The spatial components are the 'current' related to this density, such that the conservation law $\partial_\kappa J^V_\kappa = 0$ is in fact the continuity equation $\partial_t J^V_t + \partial_k J^V_k = 0$.

It is now obvious how to generalize to 3+1 dimensions. The singular field φ_{MV} has the same properties as before, and since in four dimensions the Levi-Civita symbol has four indices, our vortex current becomes an antisymmetric 2-form field,

$$J^V_{\kappa\lambda} = \epsilon_{\kappa\lambda\nu\mu}\partial_\nu\partial_\mu\varphi_{\text{MV}}. \tag{2.17}$$

The field $J^V_{\kappa\lambda}(x)$ locally represents a surface element of the vortex world sheet, defined by two non-parallel directions κ and λ. Similar as before, the temporal components J^V_{tl} are the density of vorticity of the vortex line along l. A spatial line integral around this component will result in $2\pi N$; the normal of the area enclosed by this contour is set by the two directions t and l. The purely spatial components J^V_{kl} represent the flow in the direction k of the vortex line along l. There are three independent continuity equations $\partial_\kappa J^V_{\kappa\lambda} = 0$.

This interpretation of the vortex current as field-theoretical objects will turn out to be especially useful for vortices in superconductors (chapter 4) and Mott insulators (chapter 5).

2.3 The Bose–Hubbard model

The study of quantum phase transitions concerns the collective behaviour of quantum matter at zero temperature. In many respects they resemble thermal phase transitions where one just has to replace thermal fluctuations by quantum zero-point fluctuations. Yet time plays a special role, and it is useful to consider extremely simple models that do feature the basic properties of quantum phase transitions. The simplest one would be the quantum Ising model where the dynamical variable can take only two values. One step further is to take a continuous variable and this is called the XY-model or the quantum rotor model. These systems are studied in depth in Sachdev's textbook [17]. It turns out that the latter model in the ordered state is just the quantum field theory of a free scalar field, and as such describes Goldstone modes such as the phase mode in a superfluid. The quantum phase transition arises when this phase, ordered in the superfluid, fluctuates so wildly that the long-range correlations disappear. We will see in the next section that this is equivalent to the formation of a condensate of vortices.

Here we will show how another simple model, called the Bose–Hubbard model [54], reduces to the quantum XY-model. The reason for this is twofold. Firstly, this model describes bosons hopping on a lattice but repelling each other locally. This is a realistic approximation of some real-world systems, and is in fact almost perfectly realized in cold atom experiments in optical lattices [50]. Furthermore the phase dynamics is also seen in arrays of Josephson junctions [55, 56]. The second reason is that it shows explicitly that the state across the phase transition is a Bose-Mott insulator. Therefore the disordered state after unbinding of the vortices must be equivalent to this insulating state. We will use this argument in chapter 3 to lead us to the understanding of the vortex unbinding transition in higher dimensions.

2.3.1 Bose–Hubbard Hamiltonian

We will start out from a simple Hamiltonian model for lattice bosons, and map it onto the Euclidean action of a continuum field theory, which is the most useful form for the quantum phase transition. The Hamiltonian of the Bose-Hubbard model on a D-dimensional hypercubic lattice is,

$$H_{\mathrm{BH}} = -\frac{t}{2}\sum_{\langle ij\rangle}(b_i^\dagger b_j + b_j^\dagger b_i) - \mu\sum_i n_i + \frac{U}{2}\sum_i (n_i - 1)n_i. \tag{2.18}$$

Here $b_i^\dagger$ and b_i are boson creation and annihilation operators on lattice site i, that satisfy the commutation relation $[b_i, b_j^\dagger] = \delta_{ij}$. The sum over $\langle ij\rangle$ is over nearest-neighbour sites. The number operator is $n_i = b_i^\dagger b_i$. Furthermore, the energy scales are the boson hopping t, the on-site repulsion U and the chemical potential μ. We shall assume that the chemical potential is tuned so that there is a large integer number n_0 of bosons per site. We call this "zero chemical potential". The commutation relation for n and b is,

$$[n_i, b_j] = [b_i^\dagger b_i, b_j] = 0 + [b_i^\dagger, b_j]b_i = -\delta_{ij} b_i. \tag{2.19}$$

Similarly $[n_i, b_j^\dagger] = \delta_{ij} b_i^\dagger$. To recognize quantum phase dynamics consider the substitution,

$$b_i^\dagger = \sqrt{n_i}\,\mathrm{e}^{\mathrm{i}\varphi_i}, \qquad\qquad b_i = \mathrm{e}^{-\mathrm{i}\varphi_i}\sqrt{n_i}. \tag{2.20}$$

Here φ_i is a real scalar variable. The commutation relation for n and φ follows,

$$\begin{aligned} [n_i, b_j] = \delta_{ij} b_i \qquad &\Rightarrow \qquad [n_i, \mathrm{e}^{-\mathrm{i}\varphi_j}\sqrt{n_j}] = -\delta_{ij}\mathrm{e}^{-\mathrm{i}\varphi_j}\sqrt{n_j} \\ &\Rightarrow \qquad [n_i, \mathrm{e}^{-\mathrm{i}\varphi_j}] = -\delta_{ij}\mathrm{e}^{-\mathrm{i}\phi_j}. \end{aligned} \tag{2.21}$$

This commutation relation corresponds to $[n_i, \varphi_j] = -\mathrm{i}\delta_{ij}$, which one can check via the Taylor expansion of the exponential. In this way we have switched from a description in terms of the conjugate variables b and $b^\dagger$ into the conjugate variables n and φ. Substituting this definition in Eq. (2.18) leads to,

$$H = -J\sum_{\langle ij\rangle}(1 - \cos(\varphi_i - \varphi_j)) + \frac{U}{2}\sum_i (n_i - 1)n_i. \tag{2.22}$$

Here we have defined $J = tn_0$ and added a constant term for later convenience. The physics of the weak- and strong-coupling limits is immediately clear: for large t/U, we have a superfluid where spatial fluctuations in the phase φ are very costly; for small t/U the on-site repulsion dominates, the bosons spread out evenly to minimize $U\sum_i n_i^2$ and are thereafter confined to their lattice sites: the Mott insulator.

2.3.2 Legendre transformation and continuum limit

Since we are pursuing a relativistic quantum calculation, we shall move from a Hamiltonian to a Lagrangian formalism. The commutation relation $[\varphi_i, n_j] = \mathrm{i}\delta_{ij}$ is to be compared to the canonical commutation relation $[\varphi_i, \pi_j] = \mathrm{i}\hbar\delta_{ij}$. We can therefore regard as the canonical momentum $\pi_j = \hbar n_j$. The velocity is defined by,

$$\partial_t \varphi_j = \frac{\partial H}{\partial \pi_j} = \frac{U}{\hbar^2}\pi_j. \tag{2.23}$$

From this we find the Lagrangian by Legendre transformation,

$$L = \sum_i \pi_i \partial_t \varphi_i - H = \frac{\hbar^2}{2U}\sum_i (\partial_t \varphi_i)^2 - J\sum_{\langle i,j\rangle}\big(1-\cos(\varphi_i - \varphi_j)\big), \tag{2.24}$$

which also has units of energy. Now we can take the continuum limit in D space dimensions,

$$a^D \sum_i \mapsto \int \mathrm{d}^D x, \qquad \varphi_i - \varphi_j \to a\nabla\varphi(x), \tag{2.25}$$

where a is the lattice constant. After this and expanding the cosine to leading order we find,

$$L = \frac{1}{a^D}\frac{\hbar^2}{2U}\int \mathrm{d}^D x\, (\partial_t\varphi)^2 - \frac{J}{2}\frac{1}{a^D}\int \mathrm{d}^D x\, a^2(\nabla\varphi)^2. \tag{2.26}$$

The partition function is $Z = \int \mathscr{D}\varphi\ \mathrm{e}^{\mathrm{i}/\hbar S}$, with S the action,

$$S = \int \mathrm{d}t\, L = \frac{1}{a^D}\int \mathrm{d}t\mathrm{d}^D x\ \big[\frac{\hbar^2}{2U}(\partial_t\varphi)^2 - \frac{J}{2}a^2(\nabla\varphi)^2\big]. \tag{2.27}$$

Thus, the Bose-Hubbard model at zero chemical potential is equal to the XY-model. We proceed to imaginary time $t = \mathrm{i}\tau$ to give the partition function with the Euclidean action $Z = \int \mathscr{D}\varphi\ \mathrm{e}^{-\frac{1}{\hbar}S_{\mathrm{E}}}$ where,

$$\begin{aligned} S_{\mathrm{E}} &= \frac{1}{a^D}\int \mathrm{d}\tau\mathrm{d}^D x\ \big[-\frac{\hbar^2}{2U}(\partial_\tau\varphi)^2 - \frac{J}{2}a^2(\nabla\varphi)^2\big] \\ &\equiv \int \mathrm{d}\tau\mathrm{d}^D x\ \frac{1}{2}Ja^{2-D}\ \big[-\frac{1}{c_{\mathrm{ph}}^2}(\partial_\tau\varphi)^2 - (\nabla\varphi)^2\big]. \end{aligned} \tag{2.28}$$

2.3.3 Equivalence to superfluid/Mott insulator transition

This is to be compared with the quantum action for a superfluid (cf. Eq. (3.13) in Ref. [54]),

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \, \left[-\frac{1}{2}\hbar^2 \kappa (\partial_\tau \varphi)^2 - \frac{1}{2}\hbar^2 \frac{\rho_{\mathrm{s}}}{m^*} (\nabla \varphi)^2 \right]. \tag{2.29}$$

Hence we identify the compressibility $\kappa = \frac{1}{Ua^D}$, the superfluid density divided by the boson mass $\frac{\rho_{\mathrm{s}}}{m^*} = \frac{Ja^{2-D}}{\hbar^2}$ and the superfluid velocity $c_{\mathrm{ph}} = \frac{a}{\hbar}\sqrt{UJ}$. Defining the relativistic derivative $\partial_\mu^{\mathrm{ph}} = (\frac{1}{c_{\mathrm{ph}}}\partial_\tau, \nabla)$, we find a convenient form of the action,

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \; -\frac{1}{2} J a^{2-D} \, (\partial_\mu^{\mathrm{ph}} \varphi)^2. \tag{2.30}$$

One can worry what happened to the on-site repulsion term $\sim U$? In fact, in the relativistic picture everything is contained in the fluctuations of the phase variable φ. In the superfluid the fluctuations are suppressed. But for small values of $J/U \sim J^2/c_{\mathrm{ph}}^2$, the temporal correlations $\partial_\tau \varphi$ fluctuate heavily, signalling the arbitrary creation and annihilation of vortex excitations. Thus, the destroying the superfluid takes us across the phase transition, and the disordered superfluid is equivalent to the Bose-Mott insulating state.

Indeed, this model has two stable fixed points, separated by a continuous phase transition governed by XY-universality in $D+1$ dimensions [17, 28, 32, 33, 54]. The scaling limit physics of the two stable states can be discerned by inspecting the $g \sim \sqrt{U/J} \to 0$ (weak coupling) and $g \sim \sqrt{U/J} \to \infty$ limits. In the weak coupling limit the $U(1)$ field breaks symmetry spontaneously and the theory describes the superfluid state. The small fluctuations in the phase field φ correspond either with a single Goldstone boson corresponding with the zero sound mode of the superfluid, or with the spin-wave of the quantum XY model. The strong coupling limit has an integer number of bosons n^0 per site as imposed by the choice of chemical potential. The effect of the hopping will be to create a ‘doublon’ n^0+1 and ‘holon’ n^0-1 pair on two different sites i and j: $n_i^0 n_j^0 \to (n^0-1)_i(n^0+1)_j$. This will cost an energy U: the system turns into a Bose-Mott insulator.

2.3.4 Emergent gauge invariance

The localization of the bosons implies a phenomenon that is well-known in condensed matter physics [57, 58]. This simple Mott localization has in fact

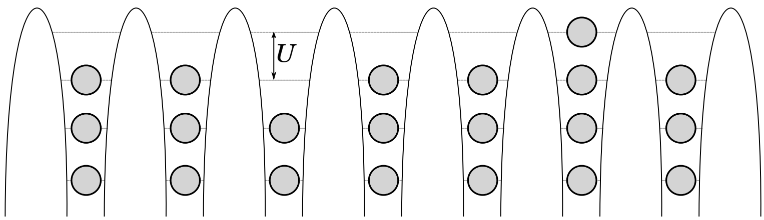


Figure 2.3: Cartoon picture of a Mott insulator. Because of the integer number of particles per site, and their strong mutual repulsion, the ground state has the same number of particles on each site. The Mott gap energy must be paid both for adding and for removing a particle. An elementary excitation is creating a doublon–holon pair; both the doublon and the holon can then propagate throughout the system without further energy penalty. This is the doublet of gapped modes.

a profound consequence: it causes a 'dynamical' emergence of a gauge symmetry. The global $U(1)$ symmetry controlling the weak coupling limit gets 'spontaneously' gauged into a compact $U(1)$ local symmetry. In the superfluid $b_i^\dagger \to \sqrt{n^0}e^{i\varphi_i}$ and the phase φ_i can undergo the global $U(1)$ symmetry transformation of the superfluid. However, in the strongly-coupled Mott insulator the number operator of the bosons is sharply quantized on every site,

$$\hat{n}_i|\Psi(\text{Mott})\rangle = n_0|\Psi(\text{Mott})\rangle \tag{2.31}$$

and this in turn implies a gauge invariance under the multiplication by an arbitrary phase α_i,

$$\begin{aligned} b_i^\dagger &\to e^{i\alpha_i}b_i^\dagger \\ b_i &\to e^{-i\alpha_i}b_i \\ \hat{n}_i &= b_i^\dagger b_i \to \hat{n}_i. \end{aligned} \tag{2.32}$$

This is the celebrated 'stay-at-home' $U(1)$ gauge invariance that has played a prominent role in the various gauge theories for high-T_c superconductivity developed for the fermionic incarnation of the Hubbard model [58]. We will return to this interesting feature in chapter 6.

2.3.5 Mode content of the Bose-Mott insulator

One can also immediately read off the nature of the collective modes of the Bose-Mott insulator from the strong coupling limit. One can either remove or add a boson and the holon and doublon that are created can just freely delocalize on the lattice giving rise to massive excitations with a mass $\simeq U/2$ given that the chemical potential is in the middle of the Mott gap (see Fig 2.3). The continuum theory we are dealing with requires that the length scales are large compared to the lattice constant, a regime that is quite different from the lattice cut-off regime exposed here. The continuum description becomes literal close to the quantum phase transition but given adiabatic continuity we know that the strong coupling limits are still representative for the mode counting and so forth. Starting close to the critical coupling on the Mott side, the Mott physics takes over from the critical regime at the correlation length (or time). At larger scales the stay-at-home gauge invariance takes over, although it now involves a volume with a dimension set by the correlation length. Accordingly, one will find the pair of degenerate propagating holon/doublon modes which appear as bound states that are pulled out of the critical continuum [31]. Similarly one finds on the superfluid side of the quantum critical point the single zero sound Goldstone boson at energies less than the scale set by the renormalized superfluid stiffness that disappears at the quantum critical point.

The simple features we have discussed in this section are generic and completely independent of the dimensionality of spacetime. Although perhaps unfamiliar, they are easily identified in the context of the standard Abelian-Higgs duality in 2+1d as discussed in the next section. In turn, they will be quite helpful in giving a firm hold in our construction of the duality in higher dimensions.

2.3.6 Charged superfluid

If we are interested in charged superfluids, i.e. superconductors, we must minimally couple to the electromagnetic potential, or photon field. Now we must recall that the gauge-covariant derivative acts on the superfluid order parameter, which is a complex scalar field $\Psi = \sqrt{\rho_{\mathrm{s}}}\mathrm{e}^{\mathrm{i}\varphi}$. Hence, the minimal coupling prescription in the London limit (ρ_{s} constant), is,

$$|\partial_\mu^{\mathrm{ph}}\Psi|^2 \to |(\partial_\mu^{\mathrm{ph}} - \mathrm{i}\frac{e^*}{\hbar}A_\mu^{\mathrm{ph}})\Psi|^2 = \rho_{\mathrm{s}}(\partial_\mu^{\mathrm{ph}}\varphi - \frac{e^*}{\hbar}A_\mu^{\mathrm{ph}})^2. \tag{2.33}$$

Here e^* is the electric charge of one boson, so of one Cooper pair. To preserve gauge invariance, the temporal component of the gauge potential should have the same velocity factor as the covariant derivative, and therefore we define $A_\mu^{\text{ph}} = (-\mathrm{i}\frac{1}{c_{\text{ph}}}V, \mathbf{A})$. Then we include the Maxwell action for the dynamics of the electromagnetic field, which is governed of course by the speed of light c. Defining the electromagnetic field tensor $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$ where $\partial_\mu = (\frac{1}{c}\partial_\tau, \nabla)$ and $A_\mu = (-\mathrm{i}\frac{1}{c}V, \mathbf{A})$, the total action is,

$$S_{\text{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \; \Big[-\frac{1}{2} J a^{2-D} \; (\partial_\mu^{\text{ph}} \varphi - \frac{e^*}{\hbar} A_\mu^{\text{ph}})^2 - \frac{1}{4\mu_0} F_{\mu\nu}^2 \Big]. \tag{2.34}$$

The identification of the dimensionful constant μ_0 as the permeability of the vacuum in units of N/A^2 is accurate only in 3+1 dimensions, but that is the case we will be mostly interested in anyway. We have established the Euclidean action of the superconductor. The equations of motion obtained by variation with respect to A_n are for instance (in real time, and substituting $Ja^{2-D} = \hbar^2 \rho / m^*$),

$$\frac{1}{c^2}\partial_t(-\partial_t A_n - \partial_n V) - \frac{1}{\mu_0}\partial_m(\partial_m A_n - \partial_n A_m) - \frac{e^* \hbar \rho}{m^*}(\partial_m \varphi - \frac{e^*}{\hbar} A_m) = 0, \tag{2.35}$$

which is one of the Ginzburg–Landau equations.

2.3.7 Dimensionless variables

It is sometimes useful to rescale all variables to be dimensionless. For our purposes this pertains especially to the charge of the dual gauge field (see next section) which has to be 1 in these dimensionless units. Starting from Eq. (2.34), we define the dimensionless variables denoted by a prime,

$$S_{\text{E}} = \hbar S'_{\text{E}}, \quad x = a x', \quad \tau = \frac{a}{c_{\text{ph}}} \tau', \quad A_m = \frac{\hbar}{e^* a} A'_m. \tag{2.36}$$

We shall suppress the primes from now on. The dimensionless version of the action Eq. (2.34) reads,

$$S_{\text{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \; -\frac{1}{2g}(\partial_\mu^{\text{ph}} \varphi - A_\mu)^2 - \frac{1}{4\mu} F_{\mu\nu}^2. \tag{2.37}$$

Here the coupling constants are,

$$\frac{1}{g} = \frac{Ja}{\hbar c_{\text{ph}}}, \qquad \frac{1}{\mu} = \frac{\hbar a^{D-3}}{\mu_0 c_{\text{ph}} e^{*2}}. \tag{2.38}$$

The first is always dimensionless, the last is dimensionless if $D = 3$, in other dimensions one has to come up with a suitable replacement for the magnetic constant μ_0. For the chargeless superfluids one lets $e^* \to 0$, which will leave only,

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \; -\frac{1}{2g}(\partial_\mu^{\mathrm{ph}} \varphi)^2. \tag{2.39}$$

2.4 Vortex duality in 2+1 dimensions

We will now perform the duality transformation of the superfluid action in 2+1 dimensions, and show how the phase transition is described as the proliferation of vortices. In 2+1 dimensions the vortices are pointlike, and trace out world lines in spacetime. Therefore their collective behaviour is captured by just a quantum field theory as for ordinary point particles. For simplicity we will proceed for the uncharged superfluid; the extension to a superconductor is straightforward by having the photon field tag along the duality transformation, the results of which are briefly mentioned at the end of this section. Here we show that vortices in a superfluid are just like charged particles with Coulomb interactions mediated by a dual gauge field. The phase transition is the proliferation of the vortices, causing the interactions to become short-ranged due to the Anderson–Higgs mechanism, which is exactly like a superconductor in this analogy.

2.4.1 Dual variables

The quantum partition sum associated with the Euclidean action Eq. (2.39) is the path integral,

$$Z = \int \mathscr{D}\varphi \; \mathrm{e}^{-\int \mathscr{L}} = \int \mathscr{D}\varphi \; \mathrm{e}^{-\int -\frac{1}{2g}(\partial_\mu^{\mathrm{ph}}\varphi)^2}. \tag{2.40}$$

For small g fluctuations of the phase φ are costly and will be much suppressed. This is the superfluid, and ϕ is the zero sound or phase mode. Even though this is already a free theory, we can still linearize for the variable φ by the introduction of an auxiliary variable w_μ through a Hubbard–Stratonovich transformation,

$$Z_{\mathrm{dual}} = \int \mathscr{D}\varphi \mathscr{D} w_\mu \; \mathrm{e}^{-\int \frac{1}{2} g w_\mu w_\mu - w_\mu \partial_\mu^{\mathrm{ph}} \varphi}, \tag{2.41}$$

The auxiliary field w_μ is a dual variable, in the sense that for this field the coupling constant is g instead of $1/g$. In canonical language going from φ

to w_μ amounts to a Legendre transform; the dual variables are in fact the canonical momenta,

$$w_\mu = -\frac{\partial \mathscr{L}}{\partial(\partial_\mu^{\text{ph}}\varphi)} = \frac{1}{g}\partial_\mu^{\text{ph}}\varphi. \tag{2.42}$$

The w_μ is also the Noether current related to the tranformation $\varphi(x) \to \varphi(x)+\alpha$ under which (2.40) is invariant. In the superfluid we identify this as the supercurrent. Integrating out the auxiliary field w_μ from Eq. (2.41) will give back the original partition sum Eq. (2.40).

2.4.2 Dual gauge field

When vortices are present in the superfluid, the otherwise smooth phase variable φ is singular inside the core region (see Fig. 2.1(b)). We therefore split it into a smooth and a multivalued part: $\varphi = \varphi_{\text{smooth}} + \varphi_{\text{MV}}$. The multivalued part denotes vortices of winding number N via,

$$\oint \mathrm{d}\varphi_{\text{MV}} = 2\pi N. \tag{2.43}$$

We have,

$$Z_{\text{dual}} = \int \mathscr{D}\varphi_{\text{MV}}\mathscr{D}\varphi_{\text{smooth}}\mathscr{D}w_\mu \;\mathrm{e}^{-\int \mathscr{L}_{\text{dual}}}, \tag{2.44}$$

$$\mathscr{L}_{\text{dual}} = \frac{1}{2}g w_\mu w_\mu - w_\mu \partial_\mu^{\text{ph}}\varphi_{\text{MV}} - w_\mu \partial_\mu^{\text{ph}}\varphi_{\text{smooth}}. \tag{2.45}$$

We can perform partial integration on the term with the smooth part of the phase field to find,

$$\mathscr{L}_{\text{dual}} = \frac{1}{2}g w_\mu w_\mu - w_\mu \partial_\mu^{\text{ph}}\varphi_{\text{MV}} - (\partial_\mu^{\text{ph}} w_\mu)\varphi_{\text{smooth}}. \tag{2.46}$$

Now we can integrate out φ_{smooth} as a Lagrange multiplier for the constraint $\partial_\mu^{\text{ph}} w_\mu = 0$. This constraint expresses the conservation of supercurrent and is in fact the continuity equation for the supercurrent $\partial_t w_t + \nabla \cdot \mathbf{w} = 0$. Thus we see that the conservation of supercurrent is due to the smoothness of the phase field. We can explicitly enforce this constraint by expressing it as the curl of a non-compact $U(1)$ gauge field,

$$w_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu^{\text{ph}} b_\lambda, \tag{2.47}$$

which is invariant under the addition of the gradient of any smooth scalar field $\varepsilon(x)$,

$$b_\lambda(x) \to b_\lambda(x) + \varepsilon(x). \tag{2.48}$$

If we substitute this into Eq. (2.46) we find,

$$Z_{\text{dual}} = \int \mathscr{D}\varphi_{\text{MV}} \mathscr{D} b_\lambda \mathscr{F}(b_\lambda)\, \mathrm{e}^{-\int \mathscr{L}_{\text{dual}}}, \tag{2.49}$$

$$\mathscr{L}_{\text{dual}} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\text{ph}}_\nu b_\lambda)^2 - \epsilon_{\mu\nu\lambda}\partial^{\text{ph}}_\nu b_\lambda \partial^{\text{ph}}_\mu \varphi_{\text{MV}}. \tag{2.50}$$

Here $\mathscr{F}(b_\lambda)$ is a gauge-fixing factor which we leave implicit from now on. Because the gauge field is smooth everywhere, we can perform integration by parts to leave,

$$\mathscr{L}_{\text{dual}} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\text{ph}}_\nu b_\lambda)^2 + b_\lambda \epsilon_{\mu\nu\lambda}\partial^{\text{ph}}_\nu \partial^{\text{ph}}_\mu \varphi_{\text{MV}} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\text{ph}}_\nu b_\lambda)^2 - b_\lambda J^{\text{V}}_\lambda. \tag{2.51}$$

Here we have defined the vortex current $J^{\text{V}}_\lambda = \epsilon_{\lambda\nu\mu}\partial^{\text{ph}}_\nu \partial^{\text{ph}}_\mu \varphi_{\text{MV}}$ as in Eq. (2.16). If we use the identity $(\epsilon_{\mu\nu\lambda}\partial^{\text{ph}}_\nu b_\lambda)^2 = \frac{1}{2}(\partial_\mu b_\nu - \partial_\nu b_\mu)^2 \equiv \frac{1}{2} f^2_{\mu\nu}$, this becomes,

$$\mathscr{L}_{\text{dual}} = \frac{1}{4} g f^2_{\nu\lambda} - b_\lambda J^{\text{V}}_\lambda. \tag{2.52}$$

This looks exactly like Maxwell electromagnetism in 2+1 dimensions, with the fluctuating dual gauge fields b_λ playing the role of the photon fields, and the vortex currents J^{V}_λ are like electrically charged monopole sources. Note that in these dimensionless units the charge of the coupling is 1. Because of this correspondence we call the superfluid in this context the Coulomb phase of the dual gauge fields. This equivalence is accidental in 2+1 dimensions, as we shall discover in the next chapter.

2.4.3 Mode content of the Coulomb phase

To see that we indeed retrieve electromagnetism for the dual fields, let us examine the two-point functions for the dual gauge field. In this context it is most convenient to go to a coordinate system in which the spatial directions are rotated to a longitudinal and a transversal component, see Fig. 1.3 on page 14. In this (τ, L, T) coordinate system, the momentum vector reads $p_\mu = (\frac{1}{c_{\text{ph}}}\omega, q, 0)$. We are free to choose the Coulomb gauge $\partial_l b_l = q b_L = 0$, such that the longitudinal component is removed. The Lagrangian for the remaining components is,

$$\mathscr{L}_{\text{dual}} = \frac{g}{2} q^2 b_\tau b_\tau + \frac{g}{2}(\frac{1}{c^2_{\text{ph}}}\omega^2 + q^2) b_T b_T - b_\tau J^{\text{V}}_\tau - b_T J^{\text{V}}_T. \tag{2.53}$$

We see that the vortex sources emit gauge fields with propagators,

$$\langle\langle b_\tau(p) b_\tau(0)\rangle\rangle = \frac{1}{gq^2}, \tag{2.54}$$

$$\langle\langle b_T(p) b_T(0)\rangle\rangle = \frac{1}{g(\frac{1}{c_{\mathrm{ph}}^2}\omega^2 + q^2)} = \frac{1}{gp^2}. \tag{2.55}$$

We recover the static long-range Coulomb force with a $\frac{1}{|\mathbf{r}|}$-potential, and the single, transversely polarized massless propagating photon of 2+1d EM, respectively. The static 'photon' reflects the well known fact that static vortices in 2D interact via a Coulomb potential, and the transversal photon is just zero sound while in the dual 'force' language it becomes explicit that this Goldstone boson can propagate forces between sources and sinks of supercurrent. We stress again that this correspondence between the 'XY universe' and 2+1d EM with scalar matter is quite accidental for the 2+1d case.

2.4.4 Vortex proliferation

The description above is suitable for one or several remote vortices in the superfluid that have long-range interactions. Upon increasing the coupling constant g, the phase fluctuations in Eq. (2.40) increase, which implies also that the spontaneous creation and annihilation of vortex–anti-vortex pairs becomes more frequent. These pairs are also longer-lived. The best description is in terms of spacetime loops of the world lines of vortex–anti-vortex pairs. The coupling constant is then as the inverse line tension, and an increasing coupling constant allows the loops to become larger and larger. At the critical point g_c the loops will have grown of the system size, and vortex lines permeating the system can freely form and disappear. This is characteristic for a condensate of particles, just as Cooper pairs can be freely extracted from the superconducting vacuum. Thus, such a "tangle of vortex world lines" is indeed equivalent to a "condensate of vortices".

This statement can be made very precise, and is in fact the central topic of Kleinert's textbooks [28, 42]. It is easiest to go to the lattice, and calculate the energy cost of meandering vortex world lines as chains of lattice links. We will not repeat this treatment here, but only cite the result, which is also established in [31, 40]. From the dual perspective it is immediately clear what will happen: the vortex condensate forms a medium (liquid) to which the dual gauge fields are minimally coupled. This just follows Ginzburg–

Landau theory of §2.1. This collective vortex condensate field is represented by a complex (dis)order parameter $\Phi = |\Phi| e^{i\phi}$, the amplitude of which corresponds to the density of the vortex fluid. The disorder parameter is related to the vortex current as,

$$J_\lambda^{\mathrm{V}} = \mathrm{i}\bar{\Phi}\partial_\lambda \Phi - \mathrm{i}(\partial_\lambda \bar{\Phi})\Phi. \tag{2.56}$$

The minimal coupling to the dual gauge field $\sim b_\lambda J_\lambda^{\mathrm{V}}$ is now reflected by the new Lagrangian,

$$\mathscr{L} = \frac{1}{2} g (\epsilon_{\mu\nu\lambda} \partial_\nu^{\mathrm{ph}} b_\lambda)^2 + \frac{1}{2} |(\partial_\lambda^{\mathrm{ph}} - \mathrm{i} b_\lambda)\Phi|^2 + \frac{\tilde{\alpha}}{2} |\Phi|^2 + \frac{\tilde{\beta}}{4} |\Phi|^4. \tag{2.57}$$

Here we have added Ginzburg–Landau potential terms. The dual gauge field b_κ clearly acts just as the electromagnetic field would in a superconductor. Thus, when $\tilde{\alpha} < 0$, the disorder parameter obtains a vacuum expectation value $|\Phi| = \sqrt{\frac{|\tilde{\alpha}|}{\tilde{\beta}}} \equiv \Phi_\infty$. Only the phase ϕ remains as a degree of freedom, it represents the density fluctuations of the vortex condensate, i.e. the compression mode of the vortex liquid.

2.4.5 Mode content of the vortex condensate

How to count the modes of the dual superconductor? It is just the usual business for the Anderson–Higgs mechanism. Choose coordinates $(\|, \perp, T)$ with $\|$ parallel to the spacetime momentum p_μ, and $\perp$ perpendicular to both $\|$ and T (Fig. 1.3). In this system the momentum becomes $p_\mu = (p, 0, 0)$. We see that the condensate phase ϕ couples only to the parallel direction,

$$\begin{aligned} \mathscr{L} &= \frac{1}{2} g (\epsilon_{\mu\nu\lambda} \partial_\nu^{\mathrm{ph}} b_\lambda)^2 + \frac{1}{2} |(\partial_\lambda^{\mathrm{ph}} - \mathrm{i} b_\lambda)\Phi|^2 \\ &\to \frac{1}{2}(p^2 + \Phi_\infty^2)(b_\perp^2 + b_T^2) + \frac{1}{2}\Phi_\infty^2 (p\phi - b_\|)^2. \end{aligned} \tag{2.58}$$

This action is invariant under the combined gauge transformations $b_\| \to b_\| + p\varepsilon$ and $\phi \to \phi + \varepsilon$. One possible gauge fix is the unitary gauge $\phi \equiv 0$ and in this way one shuffles the condensate mode into the "longitudinal photon" $b_\|$, which then becomes a true physical degree of freedom. This is sometimes referred to as the gauge field "eating the Goldstone boson". Alternatively, we can choose the Lorenz gauge $pb_\| \equiv 0$, in which this degree of freedom is indeed seen to originate in the condensate field ϕ. The field $b_\perp$ corresponds to the now short-ranged Coulomb force, and A_T and $A_\|$ form a degenerate

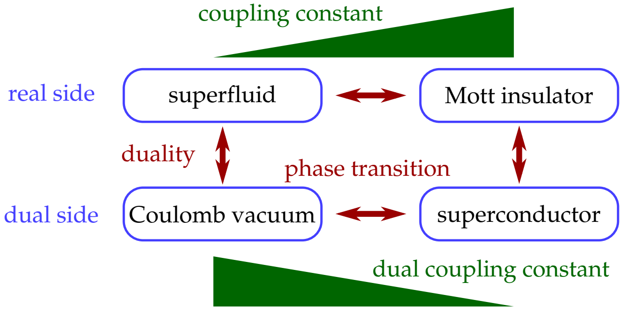


Figure 2.4: Overview of duality relations. The vertical correspondence is the duality; the horizontal is the phase transition. The dual side is in terms of the interactions between vortices: individual sources interacting via the Coulomb law; or as a superconducting condensate that effects a Higgs mechanism for the dual gauge fields. When the real coupling constant is small (the superfluid), the dual coupling constant, which is the string tension of the vortex world lines, is large and vice versa.

pair of massive propagating modes. This matches precisely the expectations that follow from the Bose-Hubbard model; in the superfluid/Coulomb phase a single massless propagating mode is present corresponding with the phase mode/photon. In the dual superconductor one finds a pair of massive propagating modes corresponding with the Higgsed transversal and longitudinal photons: these correspond with the holon and doublon excitations of the Bose-Mott insulator while the Higgs mass of the dual superconductor just codes for the Mott gap. The fate of the second mode when going to the superfluid phase was discussed in Ref. [59].

This is a good point to reflect on the correspondences in the vortex duality, see figure 2.4. The superfluid is dual to the Coulomb vacuum where the vortices take the role of the monopole charges, and the dual gauge fields are like photons. The phase transition is from the superfluid to the Bose-Mott insulator which has two gapped modes. On the dual side this is the superconductor with two massive dual photons. In duality parlance, it is sometimes said that the superfluid is dual to a superconductor; strictly speaking this is incorrect, but the since the strength of the dualities is in phase transitions, one often compares the weak-coupling phases of the dual sides.

In the next chapter we shall explore how this generalizes to higher dimensions. It turns out that not the dual gauge field but rather the supercurrent itself is the quantity containing the important information.

2.4.6 Duality squared equals unity

Just to complete the duality exercise, we can ask the question whether there can also be the analogues of Abrikosov vortices in the dual superconductor? This is indeed the case, and it goes in exactly the same way as above. First, introduce an auxiliary field v_μ, such that,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu b_\lambda)^2 + \frac{1}{2}\Phi^2_\infty(\partial^{\mathrm{ph}}_\lambda \phi - b_\lambda)^2, \tag{2.59}$$

turns into,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu b_\lambda)^2 - \frac{1}{2\Phi^2_\infty} v^2_\mu - v_\mu(\partial^{\mathrm{ph}}_\mu \phi - b_\mu). \tag{2.60}$$

If there are dual vortices, we should split the dual phase field into a smooth and a multivalued part, $\phi = \phi_{\mathrm{smooth}} + \phi_{\mathrm{MV}}$. The smooth part can be integrated out as a Lagrange multiplier for the constraint $\partial^{\mathrm{ph}}_\mu v_\mu = 0$. This constraint can be enforced explicitly by introducing yet another gauge field $v_\mu = \epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu z_\lambda$. The Lagrangian now reads,

$$\mathscr{L} = \frac{1}{2} g w^2_\mu - \frac{1}{2\Phi^2_\infty}(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu z_\lambda)^2 + z_\lambda \mathscr{J}^{\mathrm{V}}_\lambda + z_\lambda w_\lambda. \tag{2.61}$$

Here $\mathscr{J}^{\mathrm{V}}_\lambda = \epsilon_{\lambda\nu\mu}\partial^{\mathrm{ph}}_\nu \partial^{\mathrm{ph}}_\mu \phi_{\mathrm{MV}}$ is the vortex current, and we have resubstituted $w_\mu = \epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu b_\lambda$; the last term indicates how the original supercurrent couples to the z-degrees of freedom. It is at this point possible to integrate out the supercurrents w_μ, to leave a Meissner/Higgs term for the gauge fields $\frac{1}{2g} z^2_\lambda$. This indicates that the interactions between vortices $\mathscr{J}^{\mathrm{V}}_\lambda$ are Meissner screened, as it should be in a (dual) superconductor.

Instead, suppose that the vortices proliferate, then they form a condensate with order parameter Ψ, with its own Ginzburg–Landau potential,

$$\mathscr{L} = -\frac{1}{2g} z^2_\lambda - \frac{1}{2\Phi^2_\infty}(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu z_\lambda)^2 - \frac{1}{2}|(\partial_\lambda - \mathrm{i} z_\lambda)\Psi|^2 - \frac{1}{2}\alpha|\Psi|^2 - \frac{1}{4}\beta|\Psi|^4. \tag{2.62}$$

We can now rescale the gauge field $z_\lambda \to \Phi_\infty z_\lambda$, to leave,

$$\mathscr{L} = -\frac{\Phi^2_\infty}{2g} z^2_\lambda - \frac{1}{2}(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu z_\lambda)^2 - \frac{1}{2}|(\partial_\lambda - \mathrm{i}\Phi_\infty z_\lambda)\Psi|^2 - \frac{1}{2}\alpha|\Psi|^2 - \frac{1}{4}\beta|\Psi|^4. \tag{2.63}$$

The vortex condensate will destroy the dual order, with the effect that the dual superfluid density $\Phi_\infty \to 0$. In the above Lagrangian the order parameter Ψ then decouples from the dual gauge field, and we end up with just the

Landau action for a superfluid,

$$\mathscr{L} = -\frac{1}{2}|\partial_\lambda \Psi|^2 - \frac{1}{2}\alpha|\Psi|^2 - \frac{1}{4}\beta|\Psi|^4. \tag{2.64}$$

Concluding, the phase transition from the dual superconductor to *its* disordered phase is again the superfluid with which we started out. Thus indeed "duality$^2 = 1$".

Note that we have seen above that vortices can form in the dual superconductor, so there are vortices in the Bose-Mott insulator. This is a bit surprising result, that has been overlooked for quite a while. It will be a topic of interest in chapter 3 and moreover 5.

2.4.7 Charged vortex duality

For charged superfluids, i.e. superconductors, one can do the same calculation, without many changes. The starting point is the Ginzburg–Landau action Eq. (2.34), which in dimensionless units reads,

$$S_\mathrm{E} = \int \mathrm{d}\tau \mathrm{d}^D x \; -\frac{1}{2g}(\partial_\mu^\mathrm{ph}\varphi - A_\mu^\mathrm{ph})^2 - \frac{1}{4\mu}F_{\mu\nu}^2. \tag{2.65}$$

Here $1/\mu = \frac{\hbar a^{D-3}}{\mu_0 c_\mathrm{ph} e^{*2}}$. The chargeless supercurrent is defined as the canonical momentum,

$$w_\mu = -\frac{\partial \mathscr{L}}{\partial(\partial_\mu^\mathrm{ph}\varphi)} = \frac{1}{g}(\partial_\mu^\mathrm{ph}\varphi - A_\mu), \tag{2.66}$$

and is related in dimensionful units to the familiar charged supercurrent as $J_\mu^\mathrm{s} = \frac{e^*}{\hbar} w_\mu$. Separating the multivalued part of the phase field, integrating out the smooth part, and enforcing the conservation of supercurrent by introducing the dual gauge fields leads to the equivalent of Eq. (2.51),

$$\mathscr{L}_\mathrm{dual} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial_\nu^\mathrm{ph} b_\lambda)^2 - b_\lambda J_\lambda^\mathrm{V} + A_\mu \epsilon_{\mu\nu\lambda}\partial_\nu^\mathrm{ph} b_\lambda - \frac{1}{4\mu}F_{\mu\nu}^2. \tag{2.67}$$

Here we see that the photon field simply couples to the supercurrent $w_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu^\mathrm{ph} b_\lambda$ as it should. One could integrate out the dual gauge field to find an interaction between the vortex currents J_λ^V that is Meissner screened due to the electromagnetic field. But instead we proceed with the duality, where basically we just keep around the last two terms in the above expression. Thus, after proliferation of the vortices we have,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial_\nu^\mathrm{ph} b_\lambda)^2 + \frac{1}{2}\Phi_\infty^2(\partial_\lambda^\mathrm{ph}\phi - b_\lambda)^2 + A_\mu \epsilon_{\mu\nu\lambda}\partial_\nu^\mathrm{ph} b_\lambda - \frac{1}{4\mu}F_{\mu\nu}^2. \tag{2.68}$$

Here we have assumed the dual London limit $|\Phi| = \Phi_\infty$ everywhere. One could again integrate out the dual gauge field to find the electromagnetic response for the Mott insulator. We will see in §5.A.4 that we indeed find gapped poles for the conductivity instead of the delta-function response of the superconductor.

Chapter 3

Vortex duality in 3+1 dimensions

The vortex–boson or Abelian-Higgs duality as pertaining to many-body physics in 2+1 dimensions is by now well established and has been researched for over three decades [27–33, 35, 36]. One can wonder why this has almost exclusively been restricted to planar physics, while many systems of interest are in fact three-dimensional. The reason is quite simple: vortices in two dimensions are pointlike and trace out world lines, whereas in three dimensions they are linelike and trace out world sheets in spacetime. As such the dual objects are more complicated as they have more internal degrees of freedom. Although a single vortex world sheet is still quite tractable, for a rigorous description of a condensate of such extended objects, a "string foam", one needs string field theory [38, 39], which is as of yet still in early stages of development.

Surely, several authors have made progress on the condensation of vortex world sheets, in the context of string theory [47, 48] and condensed matter theory [46]. However in this chapter we shall discover that the proposed methods do not apply for the case at hand, the quantum phase transitions in 3+1 dimensional condensed matter. The reason is that they do not yield the proper mode content for the disordered phase (the Bose-Mott insulator) as they ascribe too many degrees of freedom to the vortex condensate as a compressible liquid. In part this can be explained by the fact that the vortices in condensed matter are so-called Nielsen–Olesen strings [60] which have a finite core size and core energy and no internal conformal symmetry. This is different from fundamental or 'critical' strings[1]. Nevertheless one

[1] I thank Dr Soo-Jong Rey for pointing this out.

encounters the difficulty that second quantization cannot be formulated for stringy matter. Accordingly, different from matter formed out of particles, an algorithm is lacking to compute the properties of such string condensates directly. The only example of a precise duality involving stringy topological excitations is the transversal field global Ising model in 2+1d [3]. The strong coupling phase can be viewed as Bose condensate of Ising domain walls in space time [61]; remarkably, the Wegner duality [4] demonstrates that this string condensate is actually the ordered (deconfining) phase of Ising gauge theory, while the ordered Ising phase corresponds with the confining phase of the gauged theory.

In this chapter we develop the effective theory governing the condensation of vortex world sheets in superfluids. In the ordered phase the vortices interact by exchanging 2-form gauge fields instead of 1-form or vector fields. We will show that these 2-form gauge fields undergo a Higgs mechanism in the disordered phase much like regular vector fields do. Guided by the knowledge that the disordered superfluid must correspond to the Bose-Mott insulator and its two gapped doublon and holon excitations, we argue that the string foam should add only a single degree of freedom, contrary to earlier claims. As a result, not the gauge fields but rather the physical supercurrents are to be regarded the fundamental quantities, and the phase transition is in this context at that point where supercurrents are no longer conserved. The results are generalizable to any dimension higher than two. Systems more complicated than the superfluid should undergo a similar mechanism, for instance the superconductor that will be investigated in chapter 5.

We include a discussion about vortices in the disordered phase, and two appendices on the counting of degrees of freedom and the application of this current formalism to Maxwell electromagnetism.

3.1 Dualization of the phase mode

Let us start right away by repeating as much as possible the exercise of dualizing the superfluid phase mode. The starting point is again Eq. (2.40).

$$Z = \int \mathscr{D}\varphi \ \mathrm{e}^{-\int \mathscr{L}} = \int \mathscr{D}\varphi \ \mathrm{e}^{-\int - \frac{1}{2g}(\partial_\mu^{\mathrm{ph}}\varphi)^2}. \tag{3.1}$$

Introduce auxiliary variables, the canonical momentum or the supercurrent,

$$w_\mu = -\frac{\partial \mathscr{L}}{\partial(\partial^{\rm ph}_\mu \varphi)} = \frac{1}{g}\partial^{\rm ph}_\mu \varphi. \tag{3.2}$$

The partition sum after a Hubbard–Stratonovich transformation is now,

$$Z_{\rm dual} = \int \mathscr{D}\varphi \mathscr{D} w_\mu \; {\rm e}^{-\int \frac{1}{2} g w_\mu w_\mu - w_\mu \partial^{\rm ph}_\mu \varphi}. \tag{3.3}$$

We split the phase field into smooth (phase mode) and multivalued (vortices) parts, $\varphi = \varphi_{\rm smooth} + \varphi_{\rm MV}$. In 3+1 dimensions, the contour integral around the multivalued part will still yield the winding number N times 2π, but the vortices are now linelike, because otherwise we could close the contour by pulling it 'over' the point, see §2.2. The smooth part can be integrated out as a Lagrange multiplier for the constraint $\partial^{\rm ph}_\mu w_\mu = 0$, the conservation of supercurrent.

3.1.1 2-form gauge fields

Now comes the first deviation from the treatment in 2+1 dimensions. The constraint can be explicitly enforced by expressing the supercurrent as the curl of a gauge field, but since in four dimensions the Levi-Civita symbol has four indices, the gauge field is an antisymmetric 2-form field,

$$w_\mu = \epsilon_{\mu\nu\kappa\lambda} \partial^{\rm ph}_\nu b_{\kappa\lambda}. \tag{3.4}$$

There are six independent components in $b_{\kappa\lambda}$. This expression is invariant under the addition of the gradient of any smooth vector field $\varepsilon_\lambda(x)$,

$$b_{\kappa\lambda}(x) \to b_{\kappa\lambda}(x) + \partial_\kappa \varepsilon_\lambda(x) - \partial_\lambda \varepsilon_\kappa(x). \tag{3.5}$$

The addition of the gradient of any smooth scalar field $\varepsilon_\lambda(x) \to \varepsilon_\lambda(x) + \partial_\lambda \eta(x)$ will lead to the exact same gauge transformation for $b_{\kappa\lambda}$, so there is a redundancy in the gauge redundancy itself. This is sometimes referred to as "gauge in the gauge", and is of importance in the counting of degrees of freedom as described in the appendix 3.A. The result is that a free massless 2-form field has one propagating degree of freedom, which we already know since we derived it from the superfluid phase mode. Substituting the gauge field in the generating functional we find,

$$Z_{\text{dual}} = \int \mathscr{D}\varphi_{\text{MV}} \mathscr{D} b_{\kappa\lambda} \mathscr{F}(b_{\kappa\lambda})\, \mathrm{e}^{-\int \mathscr{L}_{\text{dual}}} \tag{3.6}$$

$$\begin{aligned}\mathscr{L}_{\text{dual}} &= \frac{1}{2} g(\epsilon_{\mu\nu\kappa\lambda}\partial_\nu^{\text{ph}} b_{\kappa\lambda})^2 - \epsilon_{\mu\nu\kappa\lambda}\partial_\nu^{\text{ph}} b_{\kappa\lambda}\partial_\mu^{\text{ph}}\varphi_{\text{MV}} \\ &= \frac{1}{2} g(\epsilon_{\mu\nu\kappa\lambda}\partial_\nu^{\text{ph}} b_{\kappa\lambda})^2 - b_{\kappa\lambda} J^{\text{V}}_{\kappa\lambda}. \end{aligned} \tag{3.7}$$

Here $\mathscr{F}(b_{\kappa\lambda})$ is a suitable gauge-fixing factor, and in the last step we defined the vortex current,

$$J^{\text{V}}_{\kappa\lambda}(x) = \epsilon_{\kappa\lambda\mu\nu}\partial_\mu^{\text{ph}}\partial_\nu^{\text{ph}}\varphi_{\text{MV}}(x). \tag{3.8}$$

The interpretation of Eq. (3.7) is the following: in the superfluid there are vortex lines which trace out world sheets, built up out of surface elements $J^{\text{V}}_{\kappa\lambda}$, spanned by two non-parallel directions κ and λ. These vortices are sources in the sense of Schwinger [62], and therefore interact by exchanging two-form gauge fields $b_{\kappa\lambda}$. This gauge field corresponds to the zero sound or Goldstone boson of the superfluid. The first term is the kinetic energy or dynamics of the gauge field. Just as before, because of the long-range interactions, we call this the Coulomb phase for the vortices.

3.1.2 Mode content of the Coulomb phase

To examine the mode content explicitly, it is useful to go to the (τ, L, θ, ϕ) coordinate system, in which L is the spatial-longitudinal direction, and θ, ϕ are two arbitrarily chosen orthogonal transversal directions (see Fig. 1.3). We can use the gauge freedom Eq. (3.5) to impose the generalized Coulomb gauge $\partial_k b_{k\lambda} = q b_{L\lambda} = 0$, which removes all longitudinal components. The Lagrangian can now be expanded in the remaining components to find,

$$\mathscr{L} = \frac{1}{2} g q^2 b_{\tau\theta}^2 + \frac{1}{2} g q^2 b_{\tau\phi}^2 + \frac{1}{2} g(\omega^2 + q^2) b_{\theta\phi}^2. \tag{3.9}$$

Here we clearly identify the purely transversal component $b_{\theta\phi}$ as the single propagating mode. This makes sense as in 2+1 dimensions it was the transversal polarization of the dual gauge field, b_T, that represented the Goldstone mode. Furthermore there are now *two* temporal components $b_{\tau\theta}$ and $b_{\tau\phi}$ that communicate static Coulomb interactions between two vortex lines. The number of Coulomb forces increases because of the higher dimensionality of space: the relative orientation of vortex line sources allows for more diverse interactions. Except for this little surprise, we observe that the Coulomb phase of this stringy 2-form gauge theory is coding precisely for the physics of the 3+1d superfluid with its single propagating mode.

3.2 Vortex proliferation

Now it is time to try and increase the coupling constant g, let the vortex world sheets grow to the system size and let the vortices proliferate to effect the phase transition. We anticipate a kind of 'string foam' as the analogue of the 'tangle of vortex world lines'. As mentioned, there is presently no 'second quantized' way to do this, and all we can hope to achieve is an effective theory that captures the collective behaviour of the vortex liquid. The problem is to find a (dis)order parameter to which the dual 2-form gauge fields couple minimally. This was attempted in earlier works [46–48], and we now shall review their approach (a different path with some ideas similar to ours was taken in Refs. [49, 63]).

3.2.1 Naive generalization of the vortex proliferation

The defect world sheet is parametrized by $\sigma = (\sigma_1, \sigma_2)$ and $X(\sigma)$ is the map from the world sheet to real space. Hence each point on the world sheet σ is mapped to a specific point in real space $X(\sigma)$. A surface element of the world sheet is given by,

$$\Sigma_{\kappa\lambda}[X(\sigma)] = \frac{\partial X_\kappa}{\partial \sigma_1}\frac{\partial X_\lambda}{\partial \sigma_2} - \frac{\partial X_\lambda}{\partial \sigma_1}\frac{\partial X_\kappa}{\partial \sigma_2}. \tag{3.10}$$

The dynamics of the world sheet is given by the Nambu–Goto action,

$$S_{\text{worldsheet}} = \int \mathrm{d}^2\sigma \; T\sqrt{\Sigma_{\mu\nu}\Sigma_{\mu\nu}}, \tag{3.11}$$

where the integral is over the entire world sheet and T is the string tension, comparable to our $1/g$.

The source term $J_{\kappa\lambda} = \epsilon_{\kappa\lambda\mu\nu}\partial_\mu\partial_\nu\varphi_{\text{MV}}$ is related to the world sheet by,

$$J_{\kappa\lambda}(x) \sim \int \mathrm{d}^2\sigma \; \Sigma_{\kappa\lambda}[X(\sigma)]\delta(X(\sigma) - x). \tag{3.12}$$

According to figure 2.2(b), the gauge field $b_{\kappa\lambda}(x)$ couples to the world sheet surface element $\Sigma_{\kappa\lambda}[X(\sigma)]$. Suppose that a condensate of these vortex strings has formed, giving rise to a collective variable $\Phi[X(\sigma)]$ which is now a functional of the coordinate function $X(\sigma)$. The fluctuations of the condensate are given by the functional derivative,

$$\partial_\mu \Phi \to \frac{\delta}{\delta \Sigma_{\kappa\lambda}[X(\sigma)]} \Phi[X(\sigma)]. \tag{3.13}$$

When a condensate has formed, the amplitude $|\Phi|$ acquires a vacuum expectation value. The amplitude fluctuations freeze out as in the particle condensate and only the phase of the string condensate field is left as a dynamical variable. The phase fluctuations enumerate the collective motions of the string condensate but in the absence of an automatic formalism it is guess work to find out what these are. Marshall & Ramond, Rey and Franz [46–48] find inspiration in the analogy with the particle condensate. The phase degrees of freedom have to be matched through the covariant derivative with the 2-form gauge fields and they conjecture the seemingly obvious generalization,

$$\Phi[X(\sigma)] = |\Phi| \mathrm{e}^{\mathrm{i} \int \mathrm{d}X_\mu(\sigma) C_\mu[X(\sigma)]}, \tag{3.14}$$

which implies that the collective motions of the string condensate are parametrized in a vector valued phase. The functional derivative (3.13) yields,

$$\frac{\delta}{\delta \Sigma_{\kappa\lambda}} \Phi[X(\sigma)] = |\Phi| (\partial_\kappa C_\lambda - \partial_\lambda C_\kappa), \tag{3.15}$$

reducing in turn to a natural minimal coupling form,

$$|\frac{\delta}{\delta \Sigma_{\kappa\lambda}} \Phi| \to |(\frac{\delta}{\delta \Sigma_{\kappa\lambda}} - \mathrm{i} b_{\kappa\lambda}) \Phi| = |\Phi| (\partial_\kappa C_\lambda - \partial_\lambda C_\kappa - b_{\kappa\lambda}), \tag{3.16}$$

being gauge invariant under the combined transformations,

$$b_{\kappa\lambda} \to b_{\kappa\lambda} + \partial_\kappa \varepsilon_\lambda - \partial_\lambda \varepsilon_\kappa, \tag{3.17}$$

$$C_\kappa \to C_\kappa + \varepsilon_\kappa. \tag{3.18}$$

While this conjecture seems elegant and natural it is actually wrong, at least for the string field theory as of relevance to the 3+1d vortex string condensate. The flaw is in the overcounting of the degrees of freedom of the Mott-insulator/dual superconductor: the vector phase field ascribes too many collective degrees of freedom to the string condensate. Relying on the gauge invariance in the previous paragraph, we choose the unitary gauge $C_\kappa \equiv 0$ (cf. (2.58)). The action then reduces to that of a massive 2-form, which is known to have three propagating degrees of freedom. These can be identified by noting that we have 'spent' all gauge freedom in this gauge fix, such that all components of $b_{\kappa\lambda}$ become physical degrees of freedom. The three components $b_{\tau\lambda}$ are Coulomb forces, the other three are propagating. But we know that we should end up with two propagating degrees of freedom from the correspondence to the Bose-Mott insulator of section 2.3. Another view on this is that without interactions, this vortex condensate carries the two propagating degrees of freedom of a free massless vector field C_κ in four dimensions (just like a photon). In the unitary gauge these two get transferred to the gauge field $b_{\|\kappa}$, just as the ϕ-degree of freedom was transferred to $b_\|$ in (2.58). So if the vortex condensate were described by (3.14), it would carry two degrees of freedom, instead of only a single pressure mode.

The fallacy of this guess becomes even more obvious extending matters to higher dimensions. Generalizing this minimal coupling guess to d spacetime dimensions,

$$|\partial_\mu \phi - b_\mu| \to |\partial_{[\mu} \phi_{\nu_1 \cdots \nu_{d-3}]} - b_{\mu\nu_1\cdots\nu_{d-3}}|, \tag{3.19}$$

One easy way is to count the number of propagating degrees of freedom of the phase field $\phi_{\nu_1\cdots\nu_{d-3}}$ if it were not coupled to the gauge field $b_{\mu\nu_1\cdots\nu_{d-3}}$. All of these modes transfer to the gauge field via the Higgs mechanism, adding their degrees of freedom to the single spin-wave mode. The number of propagating modes for an antisymmetric form field is given by all possible spatial-transversal polarizations [cf. (3.9)]. In d spacetime dimensions there are $d-2$ transversal directions, which must be accommodated in the $d-3$ indices of the phase field ϕ. Therefore, the number of degrees of freedom is

$$\begin{pmatrix} d-2 \\ d-3 \end{pmatrix} = \frac{(d-2)!}{(1)!(d-3)!} = d-2, \qquad d \geq 3. \tag{3.20}$$

This must be added to the single spin-wave mode, so in d spacetime dimensions, the naive prescription (3.19) would yield $d-1$ massive degrees of freedom, overcounting the modes of the Mott insulator by $d-3$. In this regard, d=2+1 is quite special indeed!

The fact that the usual minimal coupling procedure for the Higgs phenomenon is failing so badly in the higher dimensional cases indicates that it is subtly flawed in a way that does not become obvious in the 2+1d duality case, or even the 3+1d electromagnetic Higgs condensate. What is then the correct description of the string condensate? It surely has to correspond to the Bose-Mott insulator, which implies that the string condensate can only add one additional mode. One way to establish its nature is by invoking a general physics principle: the neutral string condensate would surely represent some form of compressible quantum liquid—which is not necessarily the case for fundamental strings—and such an entity has to carry pressure and thereby a zero sound mode. There is just no room for anything else given the mode counting that we know from the Bose-Mott insulator and we can already conclude that a Nielsen–Olesen string superfluid is at macroscopic distances indistinguishable from a particle superfluid!

3.2.2 Fate of the supercurrent

We need a different approach to guide us through the phase transition. Remember that in the duality transformation, we started out with regarding the supercurrent as the central object instead of the phase mode. The supercurrent is conserved in the superfluid $\partial^{\rm ph}_\mu w_\mu = 0$, which was the reason we could express it in terms of a dual gauge field $w_\mu = \epsilon_{\mu\nu\kappa\lambda}\partial^{\rm ph}_\nu b_{\kappa\lambda}$. There is a one-to-one correspondence between the components of the supercurrent and of the gauge field when expressed in the $(\parallel,\perp,\theta,\phi)$ coordinate system,

$$w_\perp \leftrightarrow b_{\theta\phi} \qquad\qquad w_\theta \leftrightarrow b_{\perp\phi} \qquad\qquad w_\phi \leftrightarrow b_{\perp\theta}. \tag{3.21}$$

In the superfluid the conservation of supercurrent eliminates $w_\parallel$ as a degree of freedom, and for the gauge fields we can remove $b_{\parallel\lambda}\ \forall\lambda$ by a suitable gauge transformation $\partial^{\rm ph}_\kappa b_{\kappa\lambda} = 0$. This choice, called the (generalized) Lorenz gauge, is very natural as these components are not sourced by the vortex current, as it is also conserved $\partial^{\rm ph}_\kappa J^{\rm V}_{\kappa\lambda} = 0$.

But in the dual superconductor we have seen that there is an additional degree of freedom due to the vortex condensate. How is this reflected by the supercurrent?

The Helmholtz theorem, familiar from vector analysis in electrodynamics, states that a sufficiently smooth vector field can always be separated into a irrotational (curl-free) and a solenoidal (divergence-free) part. This theorem can be generalized to dimensions other than three [64]. Thus we can split any vector field, in particular the supercurrent, into,

$$w_\mu = \partial^{\rm ph}_\mu \chi + \epsilon_{\mu\nu\kappa\lambda}\partial^{\rm ph}_\nu b_{\kappa\lambda}. \tag{3.22}$$

It it easy to see that the curl of the first term and the divergence of the second term both vanish. Now in the superfluid the current is conserved, $\partial_\mu^{\mathrm{ph}} w_\mu = 0$, which imposes a constraint on the irrotational part, namely $(\partial^{\mathrm{ph}})^2 \chi = 0$. Clearly this irrotational part, corresponding to $w_\parallel$, is removed as a dynamic degree of freedom in the superfluid. But what is the situation for the vortex condensate?

Recall that the formation of a vortex line induces supercurrent to flow around it. In other words, a vortex is a source of supercurrent. In the vortex condensate vortices and anti-vortices can form and disappear freely, and as they are sources and sinks of supercurrent, the latter is no longer conserved anywhere. This is equivalent to the statement that there are now only short-range correlations of the supercurrent due to the Higgs mechanism, and the local conservation law no longer holds. The constraint $\partial_\mu^{\mathrm{ph}} w_\mu = 0$ is removed, and in view of the above this also implies the release of the irrotational, longitudinal component as an additional degree of freedom. The compressional mode of the vortex condensate is reflected by the longitudinal component of the superfluid.

3.2.3 Supercurrent Higgs action

For another viewpoint, let us step back to the 2+1 dimensional case. Using the definition $w_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu^{\mathrm{ph}} b_\lambda$ and by integrating out the phase field ϕ, Eq. (2.58) can be formally rewritten as,

$$\mathscr{L} = \frac{1}{2} g w_\mu w_\mu + \frac{1}{2} \Phi_\infty^2 w_\mu \frac{1}{-\partial^2} w_\mu. \tag{3.23}$$

Here the first term is just the kinetic energy of the supercurrent as in Eq. (2.41), and the second is the Meissner term indicative of the now short-range interactions, and it is sometimes referred to as the “gauge-invariant Higgs term”. But since this is the Higgs phase, there must also be the additional degree of freedom coming from the vortex condensate compressibility. We now know that this role is taken up by the longitudinal component of the supercurrent.

This expression is true for any dimensionality! And we have already provided the interpretation, the components of the supercurrent are classified as follows: the component $w_\perp$ corresponds to the purely transversal component of the gauge field and represent the superfluid zero sound or Goldstone mode; the transversal components w_{T_i} correspond to temporal components of the gauge fields, and therefore represent the static Coulomb forces; and the longitudinal component $w_\parallel$ couples to the vortex condensate, and is a dynamical degree of freedom only in the Higgs phase.

In light of these considerations, it is almost always best to choose the Lorenz gauge fix. Then using Eq. (3.22), the Higgs Lagrangian in 3+1 dimensions Eq. (3.23) reads,

$$\mathscr{L} = \frac{1}{2}(gp^2 + \Phi_\infty^2)(\chi^2 + b_{\perp\theta}^2 + b_{\perp\phi}^2 + b_{\theta\phi}^2). \tag{3.24}$$

Here the first two terms are the degenerate doublet of propagating modes, whereas the last two are the static Coulomb forces—their static nature with propagator $\sim q^2$ is seen only explicitly in the Coulomb gauge. All terms acquire a Higgs mass and therefore represent short-range interactions.

3.2.4 Summary of the results

The take-home message of this section is as follows. The conventional way of deriving the duality has a 'materialistic' attitude, invoking the vortices as a form of matter while the gauge fields enter much in the way as fundamental gauge fields code for the way that matter interacts. As we discussed, it is however also possible to reformulate the duality in terms of the physical currents, focussing on the way their continuity is lost—in phase representation this turns into the emergent gauge invariance of the Mott insulator. In the next section we will show that the ingredients of the vortex duality in the gauge language are strongly dependent on the dimensionality of spacetime, actually posing some problem of principle associated with the nature of string field theory.

However, when formulated in terms of the gauge invariant currents the dependence on dimensionality disappears, just as in the canonical Bose-Hubbard language of section 2.3. It leads to the correct mode counting as detailed in table 3.1. The 'current language' is still closely tied to the vortex language and this gives us the hold to control the duality in higher dimensions. The explicit statement is:

The neutral superfluid–charged superconductor duality of the 2+1d global $U(1)$ theory is equally valid in $D+1$ dimensional systems with $D > 2$, where the dual superconductor describes a $D-1$ form gauge theory Higgsed by a $p = D-2$ Nielsen–Olesen brane condensate that supports one massive compressional mode.

	Coulomb phase		Higgs phase	
	Coul. forces	propagating	Coul. forces	propagating
2+1d	1 long-range	1 massless	1 short-range	2 massive
3+1d	2 long-range	1 massless	2 short-range	2 massive

Table 3.1: Mode counting in the XY-model. Vortex proliferation in terms of the demise of supercurrents leads to the correct mode counting. Furthermore it contains as well the static Coulomb forces, which increase with the dimensionality of the system.

The derivation goes as follows. For each broken symmetry generator, there is a Goldstone mode that communicates the rigidity of that order parameter. The set of Goldstone modes $\{\varphi^a\}$ is labelled by an index a. Because these modes are massless and non-interacting, the canonical momenta $w^a_\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi^a)}$ are conserved $\partial_\mu w^a_\mu = 0$. They are in fact the Noether currents under the global symmetry transformations $\varphi^a(x) \to \varphi^a(x) + \alpha^a$. As current carries energy, the action is of the form $S \sim \int w^a_\mu w^a_\mu$. Topological defects are regions where the Goldstone variable is not well-defined; consequently, the current is no longer conserved in that region. Each flavour a of current w^a_μ can be generated by the appropriate topological defect. A condensate of such defects Φ^a will have two effects:

i) they generate current everywhere, so that the current is conserved nowhere $\partial_\mu w^a_\mu \neq 0$ which introduces a new degree of freedom;

ii) the current–current correlations are destroyed by the defects, causing them to be exponentially decay with scale set by the Higgs mass Φ^a_∞.

The action in the Higgs phase is of the form,

$$S \sim \int w^a_\mu \big(1 + \frac{(\Phi^a_\infty)^2}{-\partial^2}\big) w^a_\mu. \tag{3.25}$$

3.3 Minimal coupling to 2-form gauge fields

The Lagrangian Eq. (3.23) contains all the dynamical information, and is valid for any dimension. Still, since the gauge fields are interpreted as the force carriers of the interaction between vortices, it would be nice if there were a description in terms of the gauge fields as well. In other words, we want a minimal coupling description that supersedes Eq. (3.16), and that incorporates the 2-form gauge fields while still leading to the correct mode content. The central problem is how to match the 1-form gradient of the phase field $\partial_\mu \phi$ to the 2-form gauge field $b_{\kappa\lambda}$.

We shall here present two proposals that accomplish this task. The first is valid in any dimension, but in fact leads to a slightly different definition of the gauge field, which in turn has an effect on the vortices of the disordered phase. The second avoids this last complication, but is as of yet only valid in 3+1 dimensions, and has no obvious way in which the "duality squared"-procedure of §2.4.6 follows. Let us first describe the two proposals, and address these issues when they present themselves.

3.3.1 Orthogonal projection

Since we know that the Lagrangian in gauge field components Eq. (3.24) is correct, we would be satisfied with any minimal coupling form that results in this expression. Now this Lagrangian is explicitly gauge fixed by $\partial^{\mathrm{ph}}_\kappa b_{\kappa\lambda}$ to project out the longitudinal components. We can also collect these three components in vector form by contracting with the Levi-Civita symbol where one of the indices is fixed to be this longitudinal direction. Consequently, we propose the minimal coupling to be,

$$\partial^{\mathrm{ph}}_\mu \phi - \epsilon_{\mu\|\kappa\lambda} b_{\kappa\lambda}. \tag{3.26}$$

The second term is non-zero only when μ,κ and λ take values in $(\perp, \theta, \phi)$ exclusively. Now since the derivative operator has only a longitudinal component, any crossterms automatically vanish, and indeed we find,

$$|\Phi|^2 (\partial^{\mathrm{ph}}_\mu \phi - \epsilon_{\mu\|\kappa\lambda} b_{\kappa\lambda})^2 = |\Phi|^2 \big((\partial_\mu \phi)^2 + b^2_{\theta\phi} + b^2_{\perp\theta} + b^2_{\perp\phi} \big). \tag{3.27}$$

Several remarks are in order. Firstly, this minimal coupling does not seem to be explicitly gauge fixed, as the gauge-variant components are projected out. However after taking the square as above, one cannot help to think that the Lorenz gauge fix is still in place. This should not concern us too much: we can contend ourselves with this gauge-fixed form, knowing that the ultimate truth is represent by the "gauge-invariant Higgs action" Eq. (3.23) anyway.

Secondly and more importantly, the gauge fields $b_{\kappa\lambda}$ in this expression are not precisely the same as those we used before in e.g. Eq. (3.22). This becomes clear when we step back to 2+1 dimensions. The analogue of Eq. (3.26) is,

$$\partial_\mu \phi - \epsilon_{\mu \| \lambda} b_\lambda, \tag{3.28}$$

which is clearly different from the standard minimal coupling Eq. (2.57). In fact, the directions in the transversal directions have been shuffled by the Levi-Civita symbol. This is the reason why I refer to this minimal coupling as “orthogonal projection”. In 2+1 dimensions the relationship between the two forms for the gauge fields is clear, but in higher dimensions there is no immediate way of doing this. This does not seem to matter much now as the gauge fields are secondary variables anyway, but it has in fact bearing on the definition of the dual vortices as we will see in the next section.

Finally, this prescription can be generalized to any dimension $d \geq 2+1$,

$$\partial_\mu \phi - \epsilon_{\mu \| \lambda_1 \cdots \lambda_{d-2}} b_{\lambda_1 \cdots \lambda_{d-2}}. \tag{3.29}$$

The only surviving components of the gauge field are the single superfluid phase mode with only spatial-transversal components, and the Coulomb forces which have one index with temporal direction $\perp$.

3.3.2 Sum over vortex world sheet components

There is another form of the minimal coupling that results in Eq. (3.24), namely,

$$\frac{1}{2} \sum_\alpha \delta_{\kappa\alpha} \partial^{\mathrm{ph}}_\lambda \phi - b_{\kappa\lambda}. \tag{3.30}$$

Indeed,

$$\begin{aligned} |(\frac{1}{2} \sum_\alpha \delta_{\kappa\alpha} \partial^{\mathrm{ph}}_\lambda - b_{\kappa\lambda})\Phi|^2 &= |\Phi|^2 \big(\frac{1}{4} (\sum_\alpha \delta_{\kappa\alpha} \sum_\beta \delta_{\kappa\beta})(\partial^{\mathrm{ph}}_\lambda \phi)^2 - \sum_\alpha \delta_{\kappa\alpha} (\partial_\lambda \phi) b_{\kappa\lambda} + {b_{\kappa\lambda}}^2 \big) \\ &= |\Phi|^2 \big((\partial^{\mathrm{ph}}_\lambda \phi)^2 + {b_{\kappa\lambda}}^2 \big). \end{aligned} \tag{3.31}$$

In the last line we have imposed the Lorenz gauge so that the crossterms vanish. The expression Eq. (3.30) looks rather awkward. Nevertheless there is a concrete physical example where the minimal coupling has to be of this form, namely the vortices in a disordered superconductor. There the summation causes all κ-components of the dual vortex current $\mathscr{J}^{\mathrm{V}}_{\kappa\mu}$ to contribute to the current w_μ. This will be argued extensively in chapter 5.

Again, one could be satisfied by the correct outcome for the Lagrangian in gauge field components, always able to fall back on Eq. (3.23) when doubt arises. The specialization back to 2+1 dimensions is straightforward, by just leaving out the κ-components, avoiding the summation altogether. However it is not clear how to generalize to dimensions higher than four, but that is of no practical concern. Finally, these gauge fields $b_{\kappa\lambda}$ here are the same as used throughout this chapter, contrary to the previous construction Eq. (3.26).

3.3.3 Discussion

Exactly because the demise of the supercurrent is the defining feature of the dual Higgs condensate, there is no automatic way to derive the expression in terms of the dual gauge field. What is clear is that all of the gauge-invariant components (namely $b_{\theta\phi}$, $b_{\perp\theta}$ and $b_{\perp\phi}$) should be included and gain a Higgs mass. We are free to rotate between these components, or redefine them as we see fit. Therefore, even though the expressions Eqs. (3.26) and (3.30) look very different, we know they contain the same physics as far as the Lagrangian is concerned.

It may even be possible to define an explicit mapping between the two formulations, which would clear up the confusion that is presented here. As of yet I have not been able to find such a mapping. In the next section we will see that naively proceeding from these formulation leads to two very different interpretations of the dual vortex currents. Perhaps it is wisest to accept both forms just as different models, to be called upon in the suitable physical situation.

3.4 Vortices in the disordered phase

One of the appealing features of the vortex duality is that we have complete control over the disordered side. Indeed, in dual language it is just a Ginzburg–Landau theory of its own, with disorder parameter Φ, condensate phase fluctuations ϕ and coupling to a gauge field $b_{\kappa\lambda}$. The disordered phase is just a superconductor, albeit in 3+1 dimensions one with 2-form gauge fields.

This raises the immediate question of whether there are also dual topological defects (dual Abrikosov vortices) in the disordered phase. Since we have at hand just the theory of a (dual) superconductor, the answer is: of course there are. We already alluded to this in §2.4.6. But remembering that the disordered state is in fact the Bose-Mott insulator the appearance of such vortices is actually quite surprising. The Bose-Mott insulator is generally regarded as an exceedingly boring state of matter, where all particles are localized, everything is gapped, and there are only the two propagating doublon and holon modes. Even the dynamic spin system active in the fermionic Mott insulator is absent here.

Apparently, the state is richer and does allow for vortex excitations. For clarity I shall refer to these as Mott vortices for now on. The reason that they have not been suggested before is that usually one considers the so-called atomic or strong-coupling limit $U/t \gg 1$. But just as for superconductors, things become more interesting when the condensate is not so strong. Recall that Abrikosov vortices can appear when the penetration depth λ exceeds the coherence length, and the penetration depth is inversely proportional to the superfluid density $\lambda^2 \sim 1/|\Psi|^2$, see §2.1.2. Similarly, we expect vortices to arise in the Mott insulator when the (dis)order parameter $|\Phi|$ is not very big, so that the dual penetration depth $\tilde{\lambda}$ is large. The order parameter shrinks when one approaches the phase transition, and that would be the first place to look for them. We will have much more to say about these matters in chapter 5. Here we just show how the vortices arise in the calculation.

3.4.1 Dual vortex current

Vortices arise when there is a non-trivial winding of the dual phase field,

$$\oint \mathrm{d}\phi = \oint \mathrm{d}x^\mu \partial_\mu \phi = 2\pi N. \tag{3.32}$$

As before, we split the phase field in a smooth and a multivalued part, $\phi = \phi_{\mathrm{smooth}} + \phi_{\mathrm{MV}}$. Then we define the dual vortex current as (cf. Eq. (2.17)),

$$\mathscr{J}^{\mathrm{V}}_{\kappa\lambda} = \epsilon_{\kappa\lambda\mu\nu} \partial^{\mathrm{ph}}_\mu \partial^{\mathrm{ph}}_\nu \phi_{\mathrm{MV}}. \tag{3.33}$$

These vortices communicate via the dual currents, the fluctuations in the Mott order parameter (just as the original superfluid vortices interact via the zero sound mode). What is the nature of these vortices? The well-understood central physical quantity in all of our treatment here is the supercurrent w_μ. If we can see how the dual vortex current couples to the supercurrent, we have a clear interpretation of what the dual vortices really are.

It is possible to derive this relationship at the level of the Lagrangian, by introducing new variables that couple to the multivalued phase in the disordered phase. Then we define yet another gauge field that couples to the Mott vortices, and integrating out that gauge field will show the coupling between the Mott vortices and the original supercurrent. But we shall not take this route because i) the calculation is rather involved and yields no further insight, and ii) the current will seem to couple non-locally to the Mott vortices, while it is in fact a local coupling. It is more fruitful to simply inspect the equations of motion, and identify the physical properties from there.

3.4.2 Equation of motion: orthogonal projection

When taking the minimal coupling prescription of Eq. (3.26), the action reads,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_{\nu} b_{\kappa\lambda})^2 + \frac{1}{2}|(\partial_\mu - \mathrm{i}\epsilon_{\mu\|\kappa\lambda} b_{\kappa\lambda})\Phi|^2 + \frac{\tilde{\alpha}}{2}|\Phi|^2 + \frac{\tilde{\beta}}{4}|\Phi|^4. \tag{3.34}$$

Varying with respect to $b_{\kappa\lambda}$ leads to the equation of motion,

$$-g\epsilon_{\kappa\lambda\nu\mu}\partial^{\mathrm{ph}}_{\nu} w_\mu + \Phi^2_\infty \epsilon_{\mu\|\kappa\lambda}(\partial^{\mathrm{ph}}_{\mu}\phi - \epsilon_{\mu\|\rho\sigma} b_{\rho\sigma}) = 0. \tag{3.35}$$

Acting on this expression with the operator $\epsilon_{\alpha\beta\kappa\lambda}\partial^{\mathrm{ph}}_{\beta}$, contracting repeated indices and substituting (3.33) leads to,

$$g\partial^2 w_\mu - \Phi^2_\infty w_\mu = -\Phi^2_\infty \epsilon_{\mu\|\kappa\lambda}\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}. \tag{3.36}$$

This is to be compared to the Ginzburg–Landau expression Eq. (2.6) for the magnetic field sourced by an Abrikosov vortex. Without any vortices the right-hand side is zero, and the left-hand side indicates that the supercurrent decays exponentially over characteristic length scale $\sqrt{g/\Phi^2_\infty}$, which is the expected behaviour for a (Mott) insulating state. Conversely, a Mott vortex current $\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}$ is here a source of supercurrent locally. If we neglect the first term, this expression says that there is current wherever there is a Mott vortex.

Perhaps puzzling at first sight, this makes perfect sense: recall that a superconductor expels magnetic field, but an Abrikosov vortex consists of magnetic field permeating the superconductor through tubes, or rather vortex lines. Here the "type-II Mott insulator" expels current, but the current can penetrate locally through a vortex line.

This equation also illustrates our earlier objections to the minimal coupling prescription Eq. (3.26). On would expect that the current flows parallel to the vortex line, just as the magnetic field does in a type-II superconductor. In chapter 5 we see that this is indeed the case. However, Eq. (3.36) would set the current orthogonal to the vortex world sheet. One could argue that the vortex world sheet components are just wrongly defined, and need an additional rotation. However, one then loses the intuitive identification of the relation to the multivalued phase in real space as in Eq. (3.33). Furthermore, there is no natural way to perform this additional rotation. This form does however generalize to any higher dimension.

3.4.3 Equation of motion: sum over vortex components

When taking the minimal coupling prescription of Eq. (3.30), the action reads,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_{\nu} b_{\kappa\lambda})^2 + \frac{1}{2}|(\frac{1}{2}\sum_{\alpha}\delta_{\alpha\kappa}\partial_{\lambda} - \mathrm{i} b_{\kappa\lambda})\Phi|^2 + \frac{\tilde{\alpha}}{2}|\Phi|^2 + \frac{\tilde{\beta}}{4}|\Phi|^4. \tag{3.37}$$

Varying with respect to $b_{\kappa\lambda}$ leads to the equation of motion,

$$-g\epsilon_{\kappa\lambda\nu\mu}\partial^{\mathrm{ph}}_{\nu} w_{\mu} + \Phi^2_{\infty}\big(\frac{1}{2}\sum_{\alpha}(\delta_{\alpha\kappa}\partial_{\lambda}\phi - \delta_{\alpha\lambda}\partial_{\kappa}\phi) - b_{\kappa\lambda}\big) = 0. \tag{3.38}$$

Acting on this expression with the operator $\epsilon_{\alpha\beta\kappa\lambda}\partial^{\mathrm{ph}}_{\beta}$, contracting repeated indices and substituting (3.33) leads to,

$$g\partial^2 w_{\mu} - \Phi^2_{\infty} w_{\mu} = -\Phi^2_{\infty}\sum_{\kappa}\mathscr{J}^{\mathrm{V}}_{\kappa\mu}. \tag{3.39}$$

The left-hand side is the same as Eq. (3.36), but the right-hand side is rather different. The interpretation is as follows: a vortex line $\mathscr{J}^{\mathrm{V}}_{\kappa\mu}$ sources (super)current in the direction μ. All of the components κ contribute to this current. This may seem awkward now, but has a very natural interpretation when it represents a moving line of electric current. We will elaborate on this extensively in §5.2.

Either form of the dual vortex current, Eqs. (3.36) and (3.39), clearly couples to supercurrent. In this regard the dual vortices exactly mirror the behaviour of Abrikosov vortices in type-II superconductors: just as superconductors expel magnetic field, the Bose-Mott insulator expels supercurrent. And just as Abrikosov vortices let magnetic field permeate the superconductor in local flux lines, the dual vortices are lines of supercurrent that

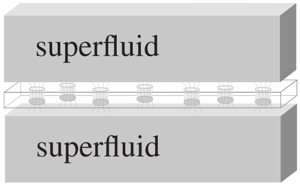


Figure 3.1: Proposed setup to show vortex lines in the Bose-Mott insulator. The Mott insulator (white) should be sandwiched between two regions with superfluid order (grey). The order parameter extends outside of the superfluid itself to pierce through the Mott insulator, in the form of vortex lines.

penetrate the insulator. Therefore we name such systems "type-II Mott insulators". The correspondence is even more striking when it is a Mott insulator made out of Cooper pairs, and that is the subject of chapter 5.

3.4.4 Tunnelling experiment

Because the superfluid is charge-neutral, the range of experimental tools that can probe these materials is limited. On the other hand, cold atoms on an optical lattice can be tuned at will to the superfluid to Mott-insulating state [50]. Furthermore, Josephson tunnelling between two superfluids has also been observed [65, 66]. Let us therefore sketch the outlines of an experiment that would create vortices in a Bose-Mott insulator.

A Josephson junction is a weak link, that can be an insulating barrier, a strip of vacuum, or just a constriction between to 'reservoirs' of superconducting order. As mentioned above, the same phenomenon has been observed in superfluids with different chemical potential. We now propose to make the barrier out of a Bose-Mott insulator near the quantum phase transition, see figure 3.1. In the regular Josephson effect, the supercurrent would flow homogeneously through the barrier, the energy cost of which grows with the volume of the barrier. But in type-II Bose-Mott insulator, the system can let the supercurrent flow through vortex lines, the energy cost of which grows with barrier width only. It is exactly like preferring the Abrikosov lattice above the fully magnetized Meissner state in type-II superconductors.

In the charged Mott insulator there is a plethora of possibilities to prove the existence of the Mott vortices, see §5.6.

3.4.5 Duality squared

For completeness, let us show that the "duality squared" procedure of §2.4.6 can also be repeated in 3+1 dimensions. As for now, I only know how to do this for the "orthogonal projection" minimal coupling prescription Eq. (3.26). But we argued that this must capture the essential physics, so we shall proceed accordingly.

We will write down only the most important steps. The minimal coupling term is linearized,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda\kappa}\partial^{\mathrm{ph}}_{\nu} b_{\kappa\lambda})^2 - \frac{1}{2}\frac{1}{\Phi^2_{\infty}} v^2_{\mu} - v_{\mu}(\partial_{\mu}\phi - \epsilon_{\mu\|\kappa\lambda} b_{\kappa\lambda}). \tag{3.40}$$

The condensate phase ϕ is split into a smooth and a multivalued part. The smooth part is integrated out to give the constraint $\partial^{\mathrm{ph}}_{\mu} v_{\mu} = 0$, which is enforced by expressing $v_{\mu} = \epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_{\nu} z_{\kappa\lambda}$. After several partial integrations and rescaling $b_{\kappa\lambda} \to \frac{1}{\sqrt{g}} b_{\kappa\lambda}$, this leads to,

$$\mathscr{L} = \frac{1}{2}(\epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_{\nu} b_{\kappa\lambda})^2 - \frac{1}{2}\frac{1}{\Phi^2_{\infty}}(\epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_{\nu} z_{\kappa\lambda})^2 + z_{\kappa\lambda}\mathscr{J}^{\mathrm{V}}_{\kappa\lambda} - \frac{1}{\sqrt{g}} z_{\kappa\lambda}\epsilon_{\kappa\lambda\mu\nu}\partial^{\mathrm{ph}}_{\nu}\epsilon_{\mu\|\rho\sigma} b_{\rho\sigma}, \tag{3.41}$$

where $\mathscr{J}^{\mathrm{V}}_{\kappa\lambda} = \epsilon_{\kappa\lambda\mu\nu}\partial^{\mathrm{ph}}_{\mu}\partial^{\mathrm{ph}}_{\nu}\phi_{\mathrm{MV}}$ is the Mott vortex current. For contractions in the last term we use the identity

$$\epsilon_{\kappa\lambda\mu\|}\epsilon_{\mu\|\rho\sigma} = \delta_{\kappa\rho}\delta_{\lambda\sigma} - \delta_{\kappa\sigma}\delta_{\lambda\rho}, \tag{3.42}$$

where the indices on the right-hand side take values orthogonal to $\|$ only. The coupling of the z-gauge field to the b-gauge field then looks like,

$$\frac{1}{\sqrt{g}} z_{\kappa\lambda}\epsilon_{\kappa\lambda\|\mu}(\epsilon_{\mu\nu\rho\sigma}\partial^{\mathrm{ph}}_{\nu} b_{\rho\sigma}) = \frac{1}{\sqrt{g}} z_{\kappa\lambda}\epsilon_{\kappa\lambda\|\mu} w_{\mu}. \tag{3.43}$$

The gauge field $b_{\rho\sigma}$ only shows up in the combination $w_{\mu} = \epsilon_{\mu\nu\rho\sigma}\partial^{\mathrm{ph}}_{\nu} b_{\rho\sigma}$, which can be integrated out to yield a Meissner term for $z_{\kappa\lambda}$,

$$\mathscr{L} = -\frac{1}{2}\frac{1}{\Phi^2_{\infty}}(\epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_{\nu} z_{\kappa\lambda})^2 - \frac{1}{2g} z^2_{\kappa\lambda} + z_{\kappa\lambda}\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}, \tag{3.44}$$

which is valid in the Lorenz gauge $\partial^{\mathrm{ph}}_{\kappa} z_{\kappa\lambda} = 0$. Here we have a theory of Abrikosov vortex strings $\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}$ that have short-range interactions with each other through the exchange of massive two-form fields $z_{\kappa\lambda}$. When vortices proliferate, they are described by a collective field Ψ, minimally coupled to the gauge field that we have rescaled $z_{\kappa\lambda} \to \Phi_\infty z_{\kappa\lambda}$,

$$\mathscr{L} = -\frac{1}{2}(\epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_\nu z_{\kappa\lambda})^2 - \frac{\Phi_\infty^2}{2g} z_{\kappa\lambda}^2 \\ - \frac{1}{2}|(\partial^{\mathrm{ph}}_\mu - \mathrm{i}\Phi_\infty \epsilon_{\mu\|\kappa\lambda} z_{\kappa\lambda})\Psi|^2 - \frac{1}{2}\alpha|\Psi|^2 - \frac{1}{4}\beta|\Psi|^4. \tag{3.45}$$

Through the phase transition, the Mott vortices destroy the dual superconducting order so that Φ_∞ vanishes. Then the gauge field $z_{\kappa\lambda}$ decouples and we are left with the action of a neutral superfluid Eq. (2.1), exactly our starting point. In this way duality$^2 = 1$ also holds in 3+1 dimensions.

3.5 Discussion

This chapter comprises the main result of this thesis: the vortex-boson duality that is so well known in condensed matter physics holds in (at least) all dimensions larger than two. The reason is that the fundamental physical quantities are the Noether currents in the ordered phase, and their conservation law imposes exactly one constraint. In the disordered phase the vortex condensate enters as a featureless fluid, whose compression mode is the additional single degree of freedom, simultaneously responsible for the demise of the currents, releasing the constraint. Related to this, all correlation functions become short-ranged due to the disorder induced by the vortices. This last statement has a very nice interpretation in terms of emergent gauge symmetry, which is the topic of chapter 6.

Even if the currents are the principal objects, the gauge fields that can be defined because of the conservation law have a natural interpretation as the force carriers of the interaction between vortices. They are the dual of the Goldstone modes. Precisely because the gauge fields couple to the vortices, they also couple minimally to the vortex condensate disorder parameter field, and are therefore instrumental in the (mathematical) construction of the dual phase transition. We have noticed that there are at least two ways to define a suitable minimal coupling, which seem equivalent at the level of the Lagrangian. But we will see in §5.2 that the precise form *is* of importance. There is room for improvement here, also considering that our proposals Eqs. (3.26) and (3.30) are not strictly gauge invariant.

These details left aside, the formalism developed here is general and should be applicable in more complex situations than the simple $U(1)$-symmetry here. The next two chapters are about charged superfluids, superconductors, in which this global symmetry is coupled to a vector gauge field, the photon. Chapter 6 alludes to its relevance for quantum liquid crystals. At the end of the thesis, chapter 7, I will contemplate some further susceptible cases.

3.A Degrees of freedom counting

We have determined the degrees of freedom by explicit examination of the action and propagators. There is a more general and formal way of deriving the *propagating* degrees of freedom given an action (Coulomb forces do not fall into this general scheme). It precisely determines the gauge degrees of freedom and the influence of constraints. This is exhaustively explained in Ref. [37]. We will very briefly discuss this procedure for free Abelian 1- and 2-forms (*op. cit.* ch.19).

The Maxwell Lagrangian in d spacetime dimensions is,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2. \tag{3.46}$$

The vector field A_μ has d components, so we start out with d degrees of freedom. The action is invariant under gauge transformation $A_\mu \to A_\mu + \partial_\mu \varepsilon$; furthermore this gauge transformation corresponds to a so-called *first-class constraint*, which means it removes two degrees of freedom in total. The reason for this is that we fix the vector field not only in space at one moment in time (a time slice), but also its evolution using $\partial_t \varepsilon$. Another point of view is that the temporal component A_t is set by the scalar electrostatic potential, which is zero everywhere for a free field; the temporal component is completely fixed by the equation of motion $\nabla^2 A_\tau = 0$.

Therefore a free vector field in d dimensions has $d-2$ propagating degrees of freedom, exactly the transversal polarizations of the photon.

The generalization of (3.46) for an anti-symmetric 2-form field $b_{\mu\nu}$ in 4 dimensions is,

$$\mathscr{L} = -\frac{1}{2}(\epsilon_{\mu\nu\kappa\lambda}\partial_\nu b_{\kappa\lambda})^2. \tag{3.47}$$

The field has six independent components. The action is invariant under gauge transformations,

$$b_{\kappa\lambda}(x) \to b_{\kappa\lambda}(x) + \partial_\kappa \varepsilon_\lambda(x) - \partial_\lambda \varepsilon_\kappa(x). \tag{3.48}$$

Here $\varepsilon_\lambda(x)$ is any smooth real vector field with 4 components; but there are only three independent gauge transformations since $\delta_{\lambda\kappa}(\partial_\kappa\varepsilon_\lambda - \partial_\lambda\varepsilon_\kappa) = 0$ always. As explained above each gauge transformation removes two degrees of freedom. The transformations are however redundant, since another vector field,

$$\varepsilon'_\lambda(x) = \varepsilon_\lambda(x) + \partial_\lambda\eta(x), \tag{3.49}$$

where η is any smooth scalar field gives exactly the same transformation in (3.48). A free 2-form field in 4 dimensions therefore has $6-(6-1) = 1$ propagating degree of freedom.

3.B Current conservation in electromagnetism

We apply the conservation-of-current considerations to the most famous example of the Higgs mechanism: the photon field in 3+1 dimensions coupled to a complex scalar condensate field. This is variously known as the Abelian–Higgs model, Ginzburg–Landau theory or scalar QED. It describes the basic physics of the electromagnetic field in the vacuum and in a superconductor.

The electromagnetic field is a vector field $A_\mu(x)$. Its dynamics is governed by the field strength $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu$ and the Maxwell action,

$$S = \int -\frac{1}{4}F_{\mu\nu}^2. \tag{3.50}$$

The field strength is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu\varepsilon$. The vector field with gauge fix $\partial_\mu A_\mu = 0$ has three degrees of freedom: the two transversal photon polarizations A_θ and A_ϕ, and the part mediating static Coulomb interactions $A_\perp$.

The field strength $F_{\mu\nu}$ has six independent components and is therefore overcounting the degrees of freedom. This can be cured by imposing the homogeneous Maxwell equations or Bianchi identities,

$$\mathsf{dF} = \epsilon_{\mu\nu\kappa\lambda}\partial_\nu F_{\kappa\lambda} = 0. \tag{3.51}$$

In $(\parallel,\perp,\theta,\phi)$-coordinates (see figure 1.3) this implies that the only non-zero components of the field strength are $F_{\parallel\nu}$, which we collect in a vector field $f_\nu \equiv F_{\parallel\nu}$ (the 'current'). From this point we act as if the field strength $F_{\parallel\nu}$ were not necessarily anti-symmetric; still the longitudinal component is set to zero as long as there are no external sources: $\partial_\nu f_\nu = \partial_\nu F_{\parallel\nu} = J_\parallel^{\text{ext}} \to 0$ (inhomogeneous

Maxwell equations). The other three components of f_ν correspond to the three physical degrees of freedom identified above via,

$$f_\nu = pA_\nu. \tag{3.52}$$

Now we couple the photon field to a complex scalar Higgs field via $|\partial_\mu \Psi| \to |(\partial_\mu - \mathrm{i}A_\mu)\Psi|$ as in (2.33). The Higgs field describes a condensate destroying the current conservation, so that the longitudinal component $f_\parallel$ is released. Indeed, from (3.52) this corresponds to the longitudinal polarization of the photon: $f_\parallel = pA_\parallel$. In terms of the field strength, it is seen to correspond to the symmetric component $F_{\parallel,\parallel}$, which is normally not taken into consideration.

Chapter 4

Electrodynamics of Abrikosov vortices

Having learned that Bose-Mott insulators near the quantum phase transition support vortex excitations, we would like to study those objects in electrically charged systems: the superconductor to charged Bose-Mott insulator transition. But first it is necessary to fully understand how the electromagnetic field comes into play, and therefore this chapter is dedicated to the vortex world sheet formalism in the superconducting state. Here the topological defects are of course the well-studied Abrikosov vortices we encountered in §2.1.2. It will prove to be an interesting subject in its own right.

The study of the matter formed from Abrikosov vortices in type-II superconductors constitutes a vast and mature research subject. This subject is crucial for the technological applications of superconductivity [67] but it has also proven to be a fertile source for fundamental condensed matter physics research. The elastic and hydrodynamical properties of matter formed from vortices can be very easily tuned by external means and it has demonstrated to be an exceedingly fertile model system to study generic questions regarding crystallization, the effects of background quenched disorder and so forth [68, 69]. Especially after the discovery of the cuprate high-T_c superconductors it became also possible to study the fluids formed from vortices. Because of the strongly two-dimensional nature of the superconductivity in the cuprates, the Abrikosov vortex lattice becomes particularly soft and it melts easily due to thermal motions at temperatures that are much below the mean field H_{c2}-line [70].

Many phenomena in this field are of a dynamical nature, associated with the fact that vortices are in motion. This includes the vortex flow, the magnetic field penetration and the flux creep, but also the large Nernst effect of the vortex fluid and, perhaps most spectacularly, the use of cuprate vortices as source of terahertz radiation [71, 72]. This vortex dynamics is analogous to the magnetohydrodynamics of electrically charged plasmas in the sense that the forces exerted on vortices are exclusively of electromagnetic origin, while in turn the vortex matter backreacts on the electromagnetic fields. The phenomena that arise are rather thoroughly understood starting from the AC and DC Josephson relations as well as the Maxwell equations as the force equations in this "vortex magnetohydrodynamics".

Although the computations explaining these phenomena are certainly correct, they are of a rather improvised, ad hoc nature, at least compared to the Landau–Lifshitz style [73] of deriving the usual magnetohydrodynamics from first principles. In this chapter we show that with the use of the vortex world sheets in 3+1 dimensional spacetime, all of the phenomena related to the electrodynamics of vortices in superconductors can be captured in one concise equation. Furthermore the electrodynamics of stringlike objects in the absence of monopole sources has very special features, turning the Maxwell field strength itself into a gauge field.

We shall show quickly how the vortex world sheet current arises in the relativistic Ginzburg–Landau model. Then we take a small theoretical detour to explore the electrodynamics of two-form sources in general. After that the rigorous vortex duality is derived for charged superfluids, and finally we shall present the equations of motion that contain all the electrodynamical phenomena related to moving Abrikosov vortices. We conclude with a short outlook.

4.1 The vortex world sheet in relativistic superconductors

We will now show how the vortex world sheet appears from the Ginzburg–Landau equations. In §4.2, we shall derive the more generic coupling of a vortex current to electromagnetic fields.

Before we write down the partition function let us stress that it may be less familiar to researchers in the field of superconductivity, since it will be fully relativistic. In particular it will have a squared time-derivative, whereas most works start with a single time-derivative term. The latter applies to systems which are diffusion-limited. Of course, in actual superconductors vortices are accompanied by such diffusion processes. However, the relativistic action is necessary to derive the vortex world sheet. Furthermore processes such as Thomas–Fermi screening are in fact ballistic. Finally the validity of this relativistic approach is verified by the results of §4.4. If one wishes to consider diffusion processes, an appropriate term can be added to the Lagrangian at will.

In this chapter we find it convenient to stay in real time, because we do not need to carry out the vortex proliferation, and because we can compare directly to other known results. The partition function associated with the relativistic Ginzburg–Landau action deep in the superconducting state is (cf. §§2.1.2,2.4.7),

$$Z = \int \mathscr{D}\varphi \mathscr{D}A_\mu \mathscr{F}(A_\mu) \mathrm{e}^{\mathrm{i}/\hbar \int \mathrm{d}^4x\, \mathscr{L}}, \tag{4.1}$$

$$\mathscr{L} = -\frac{1}{4\mu_0} F_{\mu\nu}^2 - \frac{\hbar^2}{2m^*}\rho_\mathrm{s}(\partial_\mu^\mathrm{ph}\varphi - \frac{e^*}{\hbar}A_\mu^\mathrm{ph})^2. \tag{4.2}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength; $\mathscr{F}(A_\mu)$ denotes an appropriate gauge fixing condition; φ is the superconducting phase related to the order parameter $\Psi = \sqrt{\rho_\mathrm{s}}\mathrm{e}^{\mathrm{i}\varphi}$; ρ_s is the superfluid density; m^* and e^* are the mass and charge of a Cooper pair; and most importantly, one must take great care to differentiate between the two velocities in the problem, namely the velocity of light c pertaining to the photon field A_μ, and the phase velocity in the superconductor c_ph. Therefore we have defined $\partial_\mu = (\partial_0, \nabla)$, $\partial_0 = \frac{1}{c}\partial_t$ and $\partial_\mu^\mathrm{ph} = (\partial_0^\mathrm{ph}, \nabla)$, $\partial_0^\mathrm{ph} = \frac{1}{c_\mathrm{ph}}\partial_t$. Furthermore $A_\mu = (-\frac{1}{c}V, \mathbf{A})$ and $A_\mu^\mathrm{ph} = (-\frac{1}{c_\mathrm{ph}}V, \mathbf{A})$. The last form is dictated by gauge invariance of the second term in Eq. (4.2).

We shall for the moment proceed in the relativistic limit where $c_\mathrm{ph} = c$, for simplicity. The equations of motion then follow from variation with respect to A_ν,

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathscr{L}}{\partial A_\nu} = -\frac{1}{\mu_0}\partial_\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{\hbar^2}{m^*}\rho_\mathrm{s}\frac{e^*}{\hbar}(\partial_\nu\varphi - \frac{e^*}{\hbar}A_\nu) = 0. \tag{4.3}$$

Now we act with $\epsilon_{\kappa\lambda\rho\nu}\partial_\rho$ on this equation, which leads to,

$$-\lambda^2(\epsilon_{\kappa\lambda\rho\nu}\partial_\mu^2\partial_\rho A_\nu - \cancel{\epsilon_{\kappa\lambda\rho\nu}\partial_\rho\partial_\nu}\partial_\mu A_\mu) + \epsilon_{\kappa\lambda\rho\nu}\partial_\rho A_\nu = \frac{\hbar}{e^*}\epsilon_{\kappa\lambda\rho\nu}\partial_\rho\partial_\nu\varphi = \frac{\hbar}{e^*}J^\mathrm{V}_{\kappa\lambda}. \tag{4.4}$$

Here we have defined the London penetration depth $\lambda = \sqrt{\frac{m^*}{\mu_0 e^{*2}\rho_\mathrm{s}}}$; the second term vanishes because the antisymmetric contraction of two derivatives; and on the right-hand side we recognize from Eq. (2.17) the definition of the vortex current $J^{\rm V}_{\kappa\lambda}$. Let us consider the special case $\kappa = t$, and use the definition of the magnetic field $B_l = \epsilon_{lrn}\partial_r A_n$,

$$-\lambda^2 \partial_\mu^2 B_l + B_l = \frac{\hbar}{e^*} J^{\rm V}_{tl} = \frac{\hbar}{e^*} 2\pi N \delta_l^{(2)}(\mathbf{x}). \tag{4.5}$$

Here we have used Eq. (2.17) in the last equality. This is precisely the textbook equation for the Meissner screening of a vortex source of strength N, with flux quantum $\Phi_0 = 2\pi\hbar/e^*$ Eq. (2.6), [51, eq.(5.10)]. But instead of ad hoc inserting the delta-function source, we actually derived it from the singular phase field. The only difference is that here also the dynamics is taken into account via the double time derivative contained in ∂_μ^2. The true power of the vortex world sheet shows itself when considering the electric field $\mathbf{E} = -\nabla A_0 - \partial_t \mathbf{A}$ and the spatial components $J^{\rm V}_{kl}$ of the vortex field. This will be further elaborated on in §4.4. But let us first analyze how two-form sources couple to electromagnetism in general, followed by a more general derivation of the above relations invoking a duality mapping, by which we can treat the vortex fields in the action itself, rather than only in the equations of motion. This can be regarded as revealing the more fundamental structure of the problem. The reader who is less interested in these theoretical matters may skip ahead directly to §4.4.

4.2 Electrodynamics of two-form sources

We will formulate here the generalization of the standard Maxwell action and equations of motion when the sources are not monopoles with charge density ρ and current J_m, collected in a vector field $J_\mu = (c\rho, J_m)$, but instead (vortex) lines with line densities J_{tl} and line currents J_{kl} (which denote the current in direction k of a line that extends in direction l), collected in a two-form field $J_{\kappa\lambda} = (J_{tl}, J_{kl})$. Let us first recall the established knowledge for ordinary electromagnetism, in terms suited for this generalization. For clarity we again use a shorthand notation where we are intentionally sloppy with contra- and covariant indices, leaving out dimensionful parameters in order to maximally expose the principles. In the next section we will present the final results that are accurate in this regard.

4.2.1 Maxwell action with monopole sources

Let us start by considering a set of electrical monopole sources collected in a source field J_μ as in the above, satisfying a continuity equation/conservation law $\partial_\mu J_\mu = 0$. These sources interact via the exchange of gauge particles, as gauge fields A_μ that couple locally to the source fields, by an interaction term in the Lagrangian of the form $A_\mu J_\mu$. Because of current conservation, any transformation of the gauge field $A_\mu \to A_\mu + \partial_\mu \varepsilon$, where ε is any smooth scalar field, will leave the coupling term invariant. Indeed,

$$A_\mu J_\mu \to A_\mu J_\mu + (\partial_\mu \varepsilon) J_\mu = A_\mu J_\mu - \varepsilon \partial_\mu J_\mu = A_\mu J_\mu. \tag{4.6}$$

Here we performed partial integration in the second step. The field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is also invariant under the same gauge transformation. An immediate consequence of this definition are the Bianchi identities or homogeneous Maxwell equations,

$$\epsilon_{\alpha\beta\mu\nu} \partial_\beta F_{\mu\nu} = \epsilon_{\alpha\beta\mu\nu} \partial_\beta \partial_\mu A_\nu = 0, \tag{4.7}$$

because the derivatives commute. These equations comprise $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$. This suggests a Lagrangian of gauge invariant terms,

$$\mathscr{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + A_\mu J_\mu, \tag{4.8}$$

accompanied by the Euler–Lagrange equations of motion obtained by variation with respect to A_ν,

$$\partial_\mu F_{\mu\nu} = -J_\nu. \tag{4.9}$$

These are the inhomogeneous Maxwell equations comprising $\nabla \cdot \mathbf{E} = \rho$ and $\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}$. In a superconductor, one must also add a Meissner term, which in the unitary gauge fix turns into a mass term for the gauge field A_μ,

$$\mathscr{L}_{\text{Maxwell + Meissner}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} A_\mu A_\mu + A_\mu J_\mu, \tag{4.10}$$

In this form, the Meissner term breaks the gauge invariance of the Lagrangian. This corresponds to releasing the longitudinal degrees of freedom of the photon field. A gauge equivalent perspective is that this degree of freedom represents the phase mode of the superconducting condensate (see §3.2.2). The equation of motion is modified to,

$$\partial_\mu F_{\mu\nu} - A_\nu = -J_\nu. \tag{4.11}$$

4.2.2 General two-form sources

Let us now repeat this procedure for antisymmetric two-form sources $J_{\kappa\lambda} = (J_{tl}, J_{kl})$. These must obey the continuity equations (i.e. conservation laws) $\partial_\kappa J_{\kappa\lambda} = 0$, reflecting that the density of the source can only increase (decrease) when it flows into (out of) the region under consideration, and that vortex lines cannot end within in the system (no monopoles). Consider now that these sources interact by exchanging two-form gauge fields, that we will tentatively denote by $G_{\kappa\lambda}$. Then these gauge fields couple locally to the sources as $G_{\kappa\lambda}J_{\kappa\lambda}$. These fields have to transform under gauge transformations as,

$$G_{\kappa\lambda} \to G_{\kappa\lambda} + \frac{1}{2}(\partial_\kappa \varepsilon_\lambda - \partial_\lambda \varepsilon_\kappa), \tag{4.12}$$

where ε_λ is any smooth vector field, in order to leave the coupling term invariant as required by the current conservation. Indeed,

$$G_{\kappa\lambda}J_{\kappa\lambda} \to G_{\kappa\lambda}J_{\kappa\lambda} + (\partial_\kappa \varepsilon_\lambda)J_{\kappa\lambda} = G_{\kappa\lambda}J_{\kappa\lambda} - \varepsilon_\lambda \partial_\kappa J_{\kappa\lambda} = G_{\kappa\lambda}J_{\kappa\lambda}. \tag{4.13}$$

Here we have used the antisymmetry of $J_{\kappa\lambda}$ in the first step, and partial integration in the second. The field strength $H_{\mu\kappa\lambda} = \partial_{[\mu}G_{\kappa\lambda]} = \partial_\mu G_{\kappa\lambda} + \partial_\lambda G_{\mu\kappa} + \partial_\kappa G_{\lambda\mu}$ is also invariant under these gauge transformations. An immediate consequence of this definition is the Bianchi identity,

$$\epsilon_{\nu\mu\kappa\lambda}\partial_\nu H_{\mu\kappa\lambda} = \partial_{[\nu}\partial_\mu G_{\kappa\lambda]} = 0, \tag{4.14}$$

because the derivatives commute. With these definitions, we can write down a gauge invariant Lagrangian,

$$\mathscr{L} = -\frac{1}{12}H^2_{\mu\kappa\lambda} + G_{\kappa\lambda}J_{\kappa\lambda}. \tag{4.15}$$

Note that this Lagrangian is in terms of the dynamic variables $G_{\kappa\lambda}$, which we will see later is the dual of the electromagnetic field strength $F_{\mu\nu}$. In other words, this Lagrangian is in terms of the electric and magnetic fields themselves, rather than the gauge potential A_μ. The equations of motion follow after variation with respect to $G_{\kappa\lambda}$,

$$\partial_\mu H_{\mu\kappa\lambda} = -J_{\kappa\lambda}. \tag{4.16}$$

Now, in a gauge-invariance breaking medium such as a superconductor, one must add a Higgs or Meissner term to the Lagrangian as,

$$\mathscr{L} = -\frac{1}{12}H^2_{\mu\kappa\lambda} - \frac{1}{4}G^2_{\kappa\lambda} + G_{\kappa\lambda}J_{\kappa\lambda}. \tag{4.17}$$

4.2.3 Abrikosov vortex sources

Up to now we have just reviewed the standard derivation of non-compact $U(1)$ two-form gauge theory. Let us now specialize to the case of a vortex line in a superconductor. For such an Abrikosov vortex, we know that the density J^{V}_{tl} is proportional to the magnetic field, and that the magnetic field is parallel to the spatial orientation of the vortex line. In fact, when the magnetic field intensity coincides with the lower critical field H_{c1}, the dimensionful vortex density may be denoted as before, Eq. (4.5),

$$J^{\mathrm{V}}_{tl} = \Phi_0 \delta^{(2)}_{l}(\mathbf{r}), \tag{4.18}$$

where Φ_0 is the flux quantum $\frac{h}{e^*}$. Because of these considerations, the vortex line density should couple to the magnetic field B_l. The definition of the Maxwell field strength is,

$$F_{tn} = E_n \qquad\qquad F_{mn} = \epsilon_{mnl} B_l, \tag{4.19}$$

If we contract the last definition with $\sum_{mn} \epsilon_{tbmn}$, one finds $B_l = \epsilon_{tlmn} F_{mn} \equiv G_{tl}$. Here we introduce the Hodge dual of the Maxwell field strength $G_{\alpha\beta} \equiv \frac{1}{2}\epsilon_{\alpha\beta\mu\nu} F_{\mu\nu}$. Then the coupling of the vortex line density J^{V}_{tl} to the magnetic field B_l is written as $G_{tl} J^{\mathrm{V}}_{tl}$ and generalizes to $G_{\kappa\lambda} J^{\mathrm{V}}_{\kappa\lambda}$. Therefore, the general two-form gauge field in Eq. (4.15) is now identified as the dual Maxwell field strength $G_{\kappa\lambda}$.

4.2.4 Gauge freedom of the field strength

This leads immediately to an astonishing consequence: the Maxwell field strength $F_{\mu\nu}$ itself *has now become a gauge field* ! The gauge transformations Eq. (4.12) correspond to,

$$F_{\mu\nu} \to F_{\mu\nu} + \epsilon_{\mu\nu\kappa\lambda} \partial_\kappa \varepsilon_\lambda. \tag{4.20}$$

How does it come about that these all too physical $F_{\mu\nu}$'s have suddenly turned into gauge variant quantities? The reason is simple although perhaps defeating the physical intuition: in normal matter we always have electric monopole sources J_ν with the associated equations of motion $\partial_\mu F_{\mu\nu} = -J_\nu$. In the absence of any such sources, these equations reduce to $\partial_\mu F_{\mu\nu} = 0$. Together with the inhomogeneous Maxwell equations $\epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0$, these imply that the field strength cannot be measured at all. It amounts to the

Schwinger wisdom that fields which cannot be sourced do not have physical reality [62]. The formal expression of this fact is that *the field strength becomes pure gauge in the absence of monopole sources*.

Another insight is obtained by taking a closer look at the gauge transformations Eq. (4.20). For the Bianchi identities in Eq. (4.7) these imply,

$$\begin{aligned}\epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} &\to \epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} + \epsilon_{\alpha\beta\mu\nu}\partial_\beta \epsilon_{\mu\nu\kappa\lambda}\partial_\kappa \varepsilon_\lambda \\ &= \epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} + (\partial_\alpha\partial_\lambda - \partial^2\delta_{\alpha\lambda})\varepsilon_\lambda. \end{aligned} \tag{4.21}$$

In other words, the Bianchi identities are not invariant under these gauge transformations! This makes sense: these identities are a direct result of expressing the field strength in terms of a gauge potential A_ν, which of itself has three degrees of freedom (four minus one gauge freedom). The Bianchi identities serve to restrict the six degrees of freedom contained in $F_{\mu\nu}$ to the proper number of three[1]. Conversely, in the derivation of the two-form action Eq. (4.15), we have not assumed anything about the origin of the two-form field. Next to three physical degrees of freedom, there are three gauge degrees of freedom. Therefore the constraints $\epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0$ are not strictly enforced, but can always be obtained by a suitable gauge transformation.

We never observe the gauge character of the fields $F_{\mu\nu}$ themselves because the only two-form sources to which this action applies that we know of are Abrikosov vortices in a superconductor. The superconducting matter causes a finite penetration depth λ for the fields, which is reflected by the addition of a Meissner term to the Lagrangian. The gauge-invariant form of this term is known to be,

$$H_{\kappa\lambda\mu}\frac{1}{\partial^2}H_{\kappa\lambda\mu} = -G_{\kappa\lambda}\frac{\delta_{\kappa\mu}\partial^2 - \partial_\kappa\partial_\mu}{\partial^2}G_{\kappa\lambda}, \tag{4.22}$$

in the same way as one can formally write the Meissner term in Eq. (4.10) as $F_{\mu\nu}\frac{1}{\partial^2}F_{\mu\nu}$ [cf. Eq. (3.23)]. However, since the longitudinal components of $G_{\kappa\lambda}$ are not sourced by the conserved Abrikosov vortices, we are naturally led to the Lorenz gauge condition $\partial_\kappa G_{\kappa\lambda} = 0$, and the gauge freedom has been removed. With this gauge condition Eq. (4.22) reduces to $G_{\kappa\lambda}G_{\kappa\lambda}$, that appears in Eq. (4.17). In other words, the superconducting medium forces us to the fixed frame action Eq. (4.17).

[1] In light of the discussion in §3.A, we refer here to the general case for the field strength, without restricting to a particular action. Surely a massless photon field has only two propagating degrees of freedom, but that follows only after ascertaining the Maxwell action.

4.2.5 Vortex equation of motion

We end up with the action Eq. (4.17), and we now put in dimensionful parameters. Please note that this action is equivalent to the regular action as (4.2), but with the important difference that here we work with the dual field strength $G_{\kappa\lambda}$ as the dynamic variable instead of the gauge potential A_μ .The equations of motion ("Maxwell equations for relativistic vortices") are now obtained straightforwardly by varying with respect to $G_{\kappa\lambda}$ as,

$$\lambda^2(\partial^2 G_{\mu\nu} - \partial_\mu \partial_\kappa G_{\kappa\nu} + \partial_\nu \partial_\kappa G_{\kappa\mu}) - G_{\mu\nu} = -\frac{\hbar}{e^*} J^{\mathrm{V}}_{\mu\nu}. \tag{4.23}$$

This is to be compared with Eq. (4.11) and Eq. (4.5). The second and third term can be set to zero by a gauge transformation Eq. (4.20) or alternatively by invoking the Bianchi identities Eq. (4.7). The meaning of these equations is that the two-form source $J^{\mathrm{V}}_{\kappa\lambda}$, causes an electromagnetic field $G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda} F_{\kappa\lambda}$ that is now Meissner screened over a length scale λ. The case $\mu = t$, together with the definition $B_n = \frac{1}{2}\epsilon_{nab} F_{ab}$ reduces to Eq. (4.5).

4.2.6 Summary

Summarizing, we have shown here that the action Eq. (4.17) can be postulated, from which the correct equations of motion as introduced in §4.1 directly follow, without ever mentioning the gauge potential A_μ. One trades in the Bianchi identities Eq. (4.7) for a set of gauge transformations Eq. (4.20). This action is only meaningful in the absence of monopole sources, but is very appropriate when considering two-form sources such as Abrikosov vortices. In the case that the penetration depth λ becomes infinitely large, the field strength $F_{\mu\nu}$ recovers its status as a gauge field. This would correspond to the Coulomb phase of two-form sources, as opposed to the Higgs phase that is always realized in superconductors.

As a final note it should be stressed, that although the vortex source is intrinsically dipolar in nature, the equations stated above are not generally valid for any dipole source. Here, the direction of the vortex line is always parallel to the dipole moment. If one should instead consider for instance a string of ferromagnetic material with moments not along the string, one must revert to the omnipotent regular Maxwell equations.

For the reader familiar with differential forms, I have included appendix 4.A repeating these considerations in metric-independent language, valid in any spatial dimension higher than 2.

4.3 Vortex duality in charged superfluids

We shall now rigorously derive the coupling of Abrikosov vortex sources to the electromagnetic fields, starting from the action describing a superconductor in 3+1 dimensions. This follows the same pattern as the uncharged superfluid of chapter 3, extended by minimally coupling in the electromagnetic field. For 2+1 dimensions this was done in §2.4.7. We end up with an effective action describing the electrodynamics of vortices.

4.3.1 Dual Ginzburg–Landau action

Our starting point is the partition function Eq. (4.1). To keep the equations readable, we will transform to dimensionless units denoted by a prime (which we suppress when matters are unambiguous),

$$S' = \frac{1}{\hbar}S,\ x'_m = \frac{1}{a}x_m,\ t' = \frac{c}{a}t,\ A'_\mu = \frac{ae^*}{\hbar}A_\mu,\ \rho' = \frac{\hbar a^2}{m^* c}\rho_{\mathrm{s}},\ \frac{1}{\mu'} = \frac{\hbar}{\mu_0 c e^{*2}}. \tag{4.24}$$

Here a is a length scale relevant in the system, for instance the lattice constant. We will assume the relativistic limit $c_{\mathrm{ph}} = c$; later we shall return to dimensionful quantities and it will become clear that the phase velocity is playing an essential role for the description of the non-relativistic vortices. The partition function in these dimensionless units reads,

$$Z = \int \mathscr{D}\varphi \mathscr{D}A_\mu \mathscr{F}(A_\mu) \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\ \mathscr{L}}, \tag{4.25}$$

$$\mathscr{L} = -\frac{1}{4\mu}F_{\mu\nu}^2 - \frac{1}{2}\rho(\partial_\mu\varphi - A_\mu)^2. \tag{4.26}$$

Now we perform the dualization procedure. A Hubbard–Stratonovich transformation of Eq. (4.25) leads to,

$$Z = \int \mathscr{D}w_\mu \mathscr{D}\varphi \mathscr{D}A_\mu \mathscr{F}(A_\mu) \mathrm{e}^{\mathrm{i}\int \mathscr{L}_{\mathrm{dual}}}, \tag{4.27}$$

$$\mathscr{L}_{\mathrm{dual}} = -\frac{1}{4\mu}F_{\mu\nu}^2 + \frac{1}{2\rho}w^\mu w_\mu - w^\mu(\partial_\mu\varphi - A_\mu). \tag{4.28}$$

Here w_μ is the auxiliary variable in the transformation, but it is actually the canonical momentum related to the velocity $\partial_\mu\varphi$, which can be found as,

$$w_\mu = -\frac{\partial\mathscr{L}}{\partial(\partial^\mu\varphi)} = \rho(\partial_\mu\varphi - A_\mu), \tag{4.29}$$

and is related to the supercurrent as $w_\mu = \frac{e^*}{\hbar} J_\mu$. If one integrates out the field w_μ from Eq. (4.27), one retrieves Eq. (4.25). In the presence of Abrikosov vortices, the superconductor phase φ is no longer everywhere single-valued. Therefore it is separated into smooth and multivalued parts $\varphi = \varphi_{\mathrm{smooth}} + \varphi_{\mathrm{MV}}$. The smooth part can be partially integrated yielding,

$$Z = \int \mathscr{D} w_\mu \mathscr{D}\varphi_{\mathrm{smooth}} \mathscr{D}\varphi_{\mathrm{MV}} \mathscr{D} A_\mu \mathscr{F}(A_\mu) \mathrm{e}^{\mathrm{i}\int \mathscr{L}_{\mathrm{dual}}}, \tag{4.30}$$

$$\mathscr{L}_{\mathrm{dual}} = -\frac{1}{4\mu} F_{\mu\nu}^2 + \frac{1}{\rho} w^\mu w_\mu + \varphi_{\mathrm{smooth}} \partial_\mu w^\mu - w^\mu \partial_\mu \varphi_{\mathrm{MV}} + w^\mu A_\mu. \tag{4.31}$$

Notice that the photon field is wired in just by coupling to the supercurrent. The smooth part can now be integrated out as a Lagrange multiplier turning into the constraint $\partial_\mu w^\mu = 0$, the supercurrent continuity equation. This constraint can be explicitly enforced by expressing w^μ as the curl of a gauge field,

$$w^\mu = \epsilon^{\mu\nu\kappa\lambda} \partial_\nu b_{\kappa\lambda}. \tag{4.32}$$

4.3.2 Abrikosov vortex world sheets

We can now substitute this expression in the partition function; the integral over the fields w_μ is replaced by one over $b_{\kappa\lambda}$, as long as we apply a gauge fixing term $\mathscr{F}(b_{\kappa\lambda})$ to take care of the redundant degrees of freedom. Since the gauge field is smooth it can be partially integrated to give,

$$Z = \int \mathscr{D}\varphi_{\mathrm{MV}} \mathscr{D} A_\mu \mathscr{F}(A_\mu) \mathscr{D} b_{\kappa\lambda} \mathscr{F}(b_{\kappa\lambda}) \mathrm{e}^{\mathrm{i}\int \mathscr{L}_{\mathrm{dual}}}, \tag{4.33}$$

$$\mathscr{L}_{\mathrm{dual}} = -\frac{1}{4\mu} F_{\mu\nu}^2 + \frac{1}{\rho} (\epsilon^{\mu\nu\kappa\lambda} \partial_\nu b_{\kappa\lambda})^2 - b_{\kappa\lambda} \epsilon^{\kappa\lambda\nu\mu} \partial_\nu \partial_\mu \varphi_{\mathrm{MV}} + b_{\kappa\lambda} \epsilon^{\kappa\lambda\nu\mu} \partial_\nu A_\mu. \tag{4.34}$$

Here we recognize the definition Eq. (2.17) of the vortex source,

$$J^{\mathrm{V}}_{\kappa\lambda} = \epsilon_{\kappa\lambda\nu\mu} \partial^\nu \partial^\mu \varphi_{\mathrm{MV}}, \tag{4.35}$$

and we have derived the dual partition function,

$$Z = \int \mathscr{D} J^{\mathrm{V}}_{\kappa\lambda} \mathscr{D} A_\mu \mathscr{F}(A_\mu) \mathscr{D} b_{\kappa\lambda} \mathscr{F}(b_{\kappa\lambda}) \mathrm{e}^{\mathrm{i}\int \mathscr{L}_{\mathrm{dual}}}, \tag{4.36}$$

$$\mathscr{L}_{\mathrm{dual}} = -\frac{1}{4\mu} F_{\mu\nu}^2 + \frac{1}{\rho} (\epsilon^{\mu\nu\kappa\lambda} \partial_\nu b_{\kappa\lambda})^2 - b^{\kappa\lambda} J^{\mathrm{V}}_{\kappa\lambda} + b_{\kappa\lambda} \epsilon^{\kappa\lambda\nu\mu} \partial_\nu A_\mu. \tag{4.37}$$

The interpretation is as follows. The vortex sources $J^{\mathrm{V}}_{\kappa\lambda}$ interact through the exchange of dual gauge particles $b_{\kappa\lambda}$ coding for the long range vortex–vortex interactions mediated by the condensate. The gauge field $b_{\kappa\lambda}$ couples as well to the electromagnetic field A_μ. Integrating out the electromagnetic field will lead to a Meissner/Higgs term $\sim b^2_{\kappa\lambda}$, showing that the interaction between vortices is actually short-ranged in the superconductor. However, we are instead interested in how the electromagnetic field couples to the vortices themselves. Therefore, we shall integrate out the dual gauge field $b_{\kappa\lambda}$.

The first step is to complete the square in $b_{\kappa\lambda}$. The kinetic term for $b_{\kappa\lambda}$ is proportional to,

$$-b_{\kappa\lambda}\epsilon^{\kappa\lambda\mu\nu}\partial_\nu\epsilon_{\rho\sigma\alpha\mu}\partial^\alpha b^{\rho\sigma} = -b_{\kappa\lambda}(\delta^{\kappa\mu}\partial^2 - \partial^\kappa\partial^\mu)b_{\mu\lambda} \equiv -b_{\kappa\lambda}\mathscr{G}_0^{-1\kappa\mu}b_{\mu\lambda}. \tag{4.38}$$

Here $\mathscr{G}_0^{-1\kappa\mu}$ is the inverse propagator. However, this expression cannot be inverted (the same problem arises in the quantization of the photon field). We can solve this by imposing the Lorenz gauge condition $\partial^\kappa b_{\kappa\lambda} = 0$. Then the inverse propagator is simply $\mathscr{G}_0^{-1\kappa\mu} = \delta^{\kappa\mu}\partial^2$, and its inverse is $\mathscr{G}_{0\kappa\mu} = \delta_{\kappa\mu}\frac{1}{\partial^2}$. Now we can complete the square,

$$\begin{aligned}\mathscr{L}_{\text{dual}} = &\frac{1}{2}\big(b_{\kappa\lambda} - \frac{\rho}{\partial^2}J^{\text{V}}_{\kappa\lambda} + \epsilon_{\kappa\lambda\nu\mu}\partial^\nu A^\mu\big)\big(-\frac{\partial^2}{\rho}\big)\big(b^{\kappa\lambda} - \frac{\rho}{\partial^2}J^{\text{V}\kappa\lambda} + \epsilon^{\kappa\lambda\rho\sigma}\partial_\rho A_\sigma\big)\\ &-\frac{1}{2}\big(-J^{\text{V}}_{\kappa\lambda} + \epsilon_{\kappa\lambda\nu\mu}\partial^\nu A^\mu\big)\big(-\frac{\rho}{\partial^2}\big)\big(-J^{\text{V}\kappa\lambda} + \epsilon^{\kappa\lambda\nu\mu}\partial_\nu A_\mu\big) - \frac{1}{4\mu}F^2_{\mu\nu}.\end{aligned} \tag{4.39}$$

Then we shift the field $b_{\kappa\lambda} \to b_{\kappa\lambda} + \frac{\rho}{\partial^2}J^{\text{V}}_{\kappa\lambda} - \epsilon_{\kappa\lambda\nu\mu}\partial^\nu A^\mu$ and integrate it out in the path integral to leave an unimportant constant factor. Expanding the remaining terms leads to,

$$\begin{aligned}\mathscr{L}_{\text{dual}} &= \frac{1}{2}J^{\text{V}}_{\kappa\lambda}\frac{\rho}{\partial^2}J^{\text{V}\kappa\lambda} + \frac{1}{2}\epsilon_{\kappa\lambda\nu\mu}\partial^\nu A^\mu\frac{\rho}{\partial^2}\epsilon^{\kappa\lambda\rho\sigma}\partial_\rho A_\sigma - \rho J^{\text{V}}_{\kappa\lambda}\epsilon^{\kappa\lambda\nu\mu}\frac{\partial_\nu}{\partial^2}A_\mu - \frac{1}{4\mu}F^2_{\mu\nu}\\ &= \frac{1}{2}J^{\text{V}}_{\kappa\lambda}\frac{\rho}{\partial^2}J^{\text{V}\kappa\lambda} - \frac{1}{2}\rho A^\mu A_\mu - \rho J^{\text{V}}_{\kappa\lambda}\epsilon^{\kappa\lambda\nu\mu}\frac{\partial_\nu}{\partial^2}A_\mu - \frac{1}{4\mu}F^2_{\mu\nu}.\end{aligned} \tag{4.40}$$

In going to the second line we have performed partial integration on the second term and invoked the Lorenz gauge condition $\partial^\mu A_\mu = 0$. We can immediately read off the physics encoded in this action: the first term describes the core energy of the vortices and we shall not need it in this work; the second term is the Higgs mass (including Meissner) for the electromagnetic field; the third term is the coupling term between the electromagnetic field and the vortex source. This term looks rather awkward given the derivatives in the denominator. This could signal that the coupling is non-local but that is not the case here. The origin of this coupling follows from the notions presented in section 4.2: it is not the gauge potential A_μ but rather the field strength $F_{\mu\nu}$ itself that couples to the vortex source.

4.3.3 Equations of motion

We can confirm this expectation by computing the equations of motion,

$$\frac{1}{\mu}\partial_\mu F^{\mu\nu} + \rho\epsilon^{\mu\nu\kappa\lambda}\frac{\partial_\mu}{\partial^2}J^{\mathrm{V}}_{\kappa\lambda} - \rho A^\nu = 0. \tag{4.41}$$

Acting with $\epsilon_{\alpha\beta\gamma\nu}\partial^\gamma$ on this equation, one obtains,

$$\frac{1}{\mu\rho}\epsilon_{\alpha\beta\gamma\nu}\partial^\gamma\partial_\mu F^{\mu\nu} + \epsilon_{\alpha\beta\gamma\nu}\epsilon^{\mu\nu\kappa\lambda}\frac{\partial^\gamma\partial_\mu}{\partial^2}J^{\mathrm{V}}_{\kappa\lambda} - \epsilon_{\alpha\beta\gamma\nu}\partial^\gamma A^\nu = 0 \tag{4.42}$$

Using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ one can see that from the first term only $\epsilon_{\alpha\beta\mu\nu}\partial^2 F^{\mu\nu}$ survives. Also, using $\partial^\kappa J^{\mathrm{V}}_{\kappa\lambda} = 0$ one can see that $\epsilon_{\alpha\beta\gamma\nu}\epsilon^{\mu\nu\kappa\lambda}\partial^\gamma\partial_\mu J^{\mathrm{V}}_{\kappa\lambda} = \partial^2 J^{\mathrm{V}}_{\alpha\beta}$, cancelling the derivatives in the denominator. Altogether we find,

$$\frac{1}{2\mu\rho}\epsilon_{\alpha\beta\mu\nu}\partial^2 F^{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu} = -J^{\mathrm{V}}_{\alpha\beta}. \tag{4.43}$$

This is the same result as Eq. (4.23). Notice that it is a completely local expression. As we announced earlier, we have derived here with a completely controlled procedure the dimensionless version of Eq. (4.5), describing the interactions between the vortices and electromagnetic fields inside a relativistic superconductor. Departing from this result we will derive in the next section various physical consequences. Summarizing this section, by dualizing the Ginzburg–Landau action for the superconductor, Eq. (4.25) was reformulated in terms of the vortex currents Eq. (4.35) as the active degrees of freedom, that interact via the effective gauge fields parametrizing the rigidity of the superconductor. The latter were integrated out to obtain the direct coupling of the vortices to the electromagnetic field, leading eventually to the concise equations of motion Eq. (4.43). Although this strategy is well known dealing with vortex 'particles' in 2+1 dimensions we are not aware that it was ever explored in the context of the electrodynamics of vortices in 3+1d. Surely, the derivation presented in the above is in regard with its rigour and completeness strongly contrasting with the rather ad hoc way that the problem is addressed in the standard textbooks [51, eq.(5.13)].

4.4 Vortex electrodynamics

In order to establish contact with the physics in the laboratory all that remains to be done is to break the Lorentz invariance, doing justice to the fact that the phase velocity of the superconductor as introduced in the first paragraphs of Section 4.1 is of order of the Fermi velocity of the metal and thereby a tiny fraction of the speed of light. Subsequently we will analyze what the physical ramifications are of our "Maxwell equations for vortices".

4.4.1 Non-relativistic dual action

The non-relativistic version of the vortex action Eq. (4.40) is,

$$\begin{aligned}
\mathscr{L} = & \frac{\hbar^2}{2m^*}\rho_s J^{\mathrm{V}}_{tl}\frac{1}{-1/c^2_{\mathrm{ph}}\partial_t^2+\partial_k^2}J^{\mathrm{V}}_{tl} - \frac{\hbar^2}{2m^*}\rho_s J^{\mathrm{V}}_{kl}\frac{c^2_{\mathrm{ph}}}{-1/c^2_{\mathrm{ph}}\partial_t^2+\partial_k^2}J^{\mathrm{V}}_{kl} \\
& - \frac{e^{*2}}{2m^* c^2_{\mathrm{ph}}}\rho_s V^2 - \frac{e^{*2}}{2m^*}\rho_s A_m^2 \\
& - \frac{e^*\hbar}{m^*}\rho_s \frac{1}{-\frac{1}{c^2_{\mathrm{ph}}}\partial_t^2+\partial_k^2}\Big[\frac{1}{c_{\mathrm{ph}}}J^{\mathrm{V}}_{ab}\epsilon_{abtm}(\partial_t A_m+\partial_m V)+\frac{1}{2}J^{\mathrm{V}}_{ta}\epsilon_{tamn}\partial_m A_n\Big] \\
& + \frac{1}{2\mu_0 c^2}(\partial_t A_n+\partial_n V)^2 - \frac{1}{4\mu_0}(\partial_m A_n-\partial_n A_m)^2. \qquad (4.44)
\end{aligned}$$

4.4.2 Non-relativistic equations of motion

Varying with respect to A_ν, acting with $\epsilon_{\alpha\beta\gamma\nu}\partial^\gamma$ and imposing current conservation $\partial^\kappa J^{\mathrm{V}}_{\kappa\lambda}=0$ will lead to the correct non-relativistic form of the equations of motion Eq. (4.4). However the easiest way to obtain these is to vary Eq. (4.2) directly with respect to V and A_n respectively,

$$-\frac{c^2_{\mathrm{ph}}}{c^2}\lambda^2\partial_n E_n - V = \frac{\hbar}{e^*}\partial_t\varphi, \qquad (4.45)$$

$$-\lambda^2\frac{1}{c^2}\partial_t E_n + \lambda^2\epsilon_{nmk}\partial_m B_k + A_n = \frac{\hbar}{e^*}\partial_n\varphi. \qquad (4.46)$$

Here $\lambda=\sqrt{\frac{m^*}{\mu_0 e^{*2}\rho_{\mathrm{s}}}}$ is the London penetration depth. Now we operate on the first equation by $\partial_m=\frac{1}{2}\epsilon_{mtab}\epsilon_{abrt}\partial_r$, and on the second by $\delta_{mn}\partial_t=\frac{1}{2}\epsilon_{tmab}\epsilon_{abtn}\partial_t$ and $\epsilon_{tamn}\partial_m$ respectively to obtain,

$$-\frac{c^2_{\mathrm{ph}}}{c^2}\lambda^2\partial_m\partial_n E_n - \partial_m V = \frac{\hbar}{e^*}c_{\mathrm{ph}}\frac{1}{2}\epsilon_{mtab}J^{\mathrm{V}}_{ab}, \qquad (4.47)$$

$$-\lambda^2\frac{1}{c^2}\partial_t^2 E_m - \lambda^2\partial_n^2\partial_t A_m + \partial_t A_m = \frac{\hbar}{e^*}c_{\mathrm{ph}}\frac{1}{2}\epsilon_{tmab}J^{\mathrm{V}}_{ab}, \qquad (4.48)$$

$$\lambda^2(\nabla^2-\frac{1}{c^2}\partial_t^2)B_a - B_a = -\frac{\hbar}{e^*}J^{\mathrm{V}}_{ta}. \qquad (4.49)$$

For the last equation we used the Maxwell equations $\nabla\times\mathbf{E} = -\partial_t\mathbf{B}$ and $\nabla\cdot\mathbf{B} = 0$. This one is equal to the one we found before in Eq. (4.5), obviously, since there the temporal terms do not come into play.

For the equations for the electric field is it useful to choose the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$, and separate the electric field in longitudinal and transversal parts: $\mathbf{E} = \mathbf{E}^{\mathrm{L}} + \mathbf{E}^{\mathrm{T}}$, where $\nabla\times\mathbf{E}^{\mathrm{L}} = 0$ and $\nabla\cdot\mathbf{E}^{\mathrm{T}} = 0$. In the Coulomb gauge we see from the definition $\mathbf{E} = -\nabla V - \partial_t\mathbf{A}$ that $\mathbf{E}^{\mathrm{L}} = -\nabla V$ and $\mathbf{E}^{\mathrm{T}} = -\partial_t\mathbf{A}$. We can subtract the first equation above from the second to obtain,

$$\lambda^2\big(-\frac{1}{c^2}\partial_t^2 E_m + \nabla^2 E_m^{\mathrm{T}} + \frac{c_{\mathrm{ph}}^2}{c^2}\nabla^2 E_m^{\mathrm{L}}\big) - E_m = \frac{\hbar}{e^*}c_{\mathrm{ph}}\epsilon_{tmab}J^{\mathrm{V}}_{ab}. \tag{4.50}$$

Hence, as in the case of the Maxwell theory for non-relativistic matter one finds instead of the highly symmetric relativistic result Eq. (4.4) two equations of motion that are representing the spatial (magnetic) and temporal (electrical) sides of the physics, Eq. (4.49) and Eq. (4.50). One notices that the first 'magnetic' equation is quite like the relativistic one while the 'electrical' equation is now more complicated for reasons that will become clear in a moment.

The factor c_{ph} on the right-hand side of the electric equation is due to our convention of rescaling the time derivative to having units of 1/length in the definition of $J^{\mathrm{V}}_{\kappa\lambda}$. Thus all components of $J^{\mathrm{V}}_{\kappa\lambda}$ have dimensions of a surface density, and multiplying by a velocity is necessary to end up with a current density. The sign difference on the right-hand side between the electric and magnetic equations is related to the continuity equation $\frac{1}{c_{\mathrm{ph}}}\partial_t J^{\mathrm{V}}_{tn} = -\partial_m J^{\mathrm{V}}_{mb}$.

To grasp the content of these equations, one should compare the magnetic equation Eq. (4.49) with the standard form [51, eq.(5.13)],

$$\lambda^2\nabla^2 B_a - B_a = -\Phi_0\delta_a^{(2)}(\mathbf{r}), \tag{4.51}$$

Here $\Phi_0 = 2\pi\hbar/e^*$ is the flux quantum. The factor of 2π is associated with the definition of J^{V} as in Eq. (2.17). Our treatment automatically takes dynamics into account in the form of temporal derivatives. Otherwise, the correspondence is complete. We have indeed exactly recovered the well-established vortex equation of motion.

The equation for the electric field (4.50) looks more involved, but this can be made more insightful by writing the equations for the longitudinal and transversal parts separately,

$$\lambda^2(\frac{c_{\text{ph}}^2}{c^2}\nabla^2 - \tfrac{1}{c^2}\partial_t^2)E_m^{\text{L}} - E_m^{\text{L}} = \frac{\hbar}{e^*}c_{\text{ph}}\epsilon_{tmab}^{\text{L}}J_{ab}^{\text{V}}, \tag{4.52}$$

$$\lambda^2(\nabla^2 - \tfrac{1}{c^2}\partial_t^2)E_m^{\text{T}} - E_m^{\text{T}} = \frac{\hbar}{e^*}c_{\text{ph}}\epsilon_{tmab}^{\text{T}}J_{ab}^{\text{V}}. \tag{4.53}$$

The labels on the ϵ-symbol denote that they include a longitudinal or transversal projection.

We want to point out for future reference that, applying the curl operator to Eq. (4.49), in the absence of vortex sources, and using $\nabla \times \mathbf{B} = -\mu_0 \mathbf{J}$ (the Ampère–Maxwell equation in the static limit), one finds,

$$\lambda^2\nabla^2\mathbf{J} - \mathbf{J} = 0. \tag{4.54}$$

This denotes the perhaps counterintuitive result that the current is screened inside the superconductor. The reason is that a current induces a magnetic field locally, and the superconductor wants to expel the magnetic field. As such, all current through a superconductor flows through a thin layer near the boundary of typical size λ.

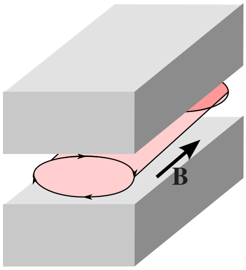
4.4.3 Vortex phenomenology

We can now read off the following physical relations:

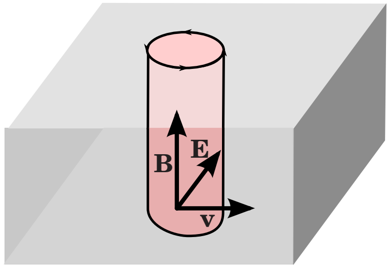
1. *Meissner screening*: from Eq. (4.49) in the static limit $\partial_t \to 0$, a vortex line sources a magnetic field, that falls off in the superconductor with a length scale λ, the familiar Meissner effect.

2. *Thomas–Fermi screening*: from Eq. (4.52) one infers that the longitudinal (electrostatic) electric field penetrates up to a much smaller length $\frac{c_{\text{ph}}}{c}\lambda$, which is the Thomas–Fermi length ($c \approx 300 c_{\text{ph}}$). This just amounts to the well-known fact that the electrical screening is the same in the metal as in the superconductor. Notice that this length scale is obtained without referral to the electrons in the normal metal state as in the textbook derivation.

3. *Dynamic Meissner screening or the Higgs mass*: taking into account the time-dependence, Eq. (4.49) and Eq. (4.53) show that the transversal photon parts of the fields are screened not only in space, but also in time with characteristic time scale $\frac{\lambda}{c}$. This is just the familiar statement that the two propagating photon polarizations in 3+1 dimensions acquire a "Higgs mass" $\sim \frac{\hbar}{\lambda c}$ inside the superconductor.



(a) A vortex in a Josephson junction between two superconductors (grey); it has no normal core. The magnetic field **B** is along the vortex; any electric field across the junction causes the vortex to move in the perpendicular direction. Such motion induces electromagnetic radiation that may escape to the outside world.



(b) Geometry of the electric field **E** generated by a vortex line parallel to the magnetic field **B** and moving with a speed **v**. This phenomenon related to the Lorentz force follows directly from the vortex equations of motion.

Figure 4.1: Additional vortex configurations

4. *Electrical field of a moving vortex and the Nernst effect*: disregarding the dynamical term in Eq. (4.50), one is left with

$$E_m = -\frac{\hbar}{e^*} c_{\text{ph}} \epsilon_{tmkl} J^{\text{V}}_{kl}. \tag{4.55}$$

Recall from section 2.2.4 that we had interpreted J^{V}_{kl} as the flow or velocity in the k-direction of a vortex line in the l direction. Since we know that one vortex line carries a magnetic flux of $\Phi_0 = 2\pi \frac{\hbar}{e^*}$, we can write $\frac{\hbar}{e^*} c_{\text{ph}} J^{\text{V}}_{kl} = v_l B^0_k$, where B^0 denotes the field associated with one quantum of flux, and $v_l = c_{\text{ph}} \hat{e}_l$ is the velocity. In practice there is always a drag force that greatly slows down the vortices. Still, Josephson vortices that do not have a normal core (Fig. 4.1(a)) may achieve this large speed. With this interpretation, (4.50) reads,

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}^0, \tag{4.56}$$

which is precisely the known result [74] for the electric field generated by a vortex moving in a magnetic field B^0 (Fig. 4.1(b)). When the motion is caused by a temperature gradient this is responsible for the large Nernst effect of the vortex fluid.

5. *AC Josephson relation*: another interpretation of Eq. (4.52) is found by inserting the definition of the vortex current, $J^{\text{V}}_{ab} = \epsilon_{abtn} \frac{1}{c_{\text{ph}}} \partial_t \partial_n \varphi$, taking

m as the longitudinal direction and neglecting the higher derivative terms. In this case,

$$\partial_m V = \frac{\hbar}{e^*}\partial_t(\partial_m \varphi). \tag{4.57}$$

Here the left-hand side is the potential difference, and the right-hand side is the time derivative of the superconducting phase difference. This is exactly the AC Josephson relation. The full equations Eq. (4.50) reveal also that the induced electric field is screened inside the superconductor.

6. *Moving vortices as radiation sources*: in the same spirit, the moving vortex is also inducing dynamic transversal fields according to Eq. (4.53). In other words: moving vortices radiate [71]. But since the field is Meissner screened, it is very hard to detect this radiation. All our results also apply to Josephson vortices (line vortex solutions in a Josephson junction between two superconductors parallel to the interface, Fig. 4.1(a)), which differ only in the regard that they do not have a normal core. There is much recent interest in radiation from (arrays of) Josephson junctions, see e.g. [75]. Since inside the junction the field is not expelled by Meissner and metallic screening, the radiation may escape to the outside world. In this literature one finds the following result [76, eq.(13)],

$$-\hat{\lambda}^2\nabla^2\mathbf{A} + \mathbf{A} = \frac{\hbar}{e^*}\nabla\phi. \tag{4.58}$$

Here $\hat{\lambda}$ differs from λ because of a special geometry. Compare this with a result that follows from Eq. (4.53),

$$\partial_t\big[-\lambda^2\big((\nabla^2 - \tfrac{1}{c^2}\partial_t^2)\mathbf{A} + \mathbf{A}\big] = \partial_t\big[\frac{\hbar}{e^*}\nabla\varphi\big], \tag{4.59}$$

confirming Eq. (4.58) but showing in addition how to take care of a possible time dependence of the photon field.

Summarizing, to the best of my knowledge we have addressed all known electrodynamical properties of vortex matter departing from a single action principle.

4.5 Outlook

I am of the opinion that our action principle for vortex electrodynamics Eq. (4.40) resp. (4.44) and the associated "vortex-Maxwell" equations Eq. (4.43), (4.49) and (4.50) deserve a place in the textbooks on the subject. In contrast with the clever but improvising discussions one usually finds, our formulation has the same 'mechanical' quality as for instance the Landau–Lifshitz treatise of electromagnetism. One just departs from the fundamentals, to expose the consequences by unambiguous and straightforward algebraic manipulations that are worshipped by any student of physics. A potential hurdle is that one has to get familiar with the two-form gauge field formalism, but then again this belongs to the kindergarten of differential geometry and string theory.

The analysis also reveals the origin of the peculiar nature of this vortex electrodynamics. The realization that it is in fact governed by a two-form gauge structure amounts to an entertaining excursion in the fundamentals of gauge theory itself, nota bene associated with the superficially rather mundane and technology-oriented vortex physics, at least when viewed from the perspective of fundamental physics. In the next chapter we will encounter more surprises when we investigate the electrodynamics of vortices in Bose-Mott insulators

On the practical side, as we implicitly emphasized in the last section our approach offers a unified description of the electrodynamics of vortices. Although we got as far as recovering the known physical effects in terms of special limits of our equations, there is potential to use them to identify hitherto unknown effects and perhaps to arrive at a more complete description of the electrodynamics vortex matter. As we are well aware of the large body of knowledge of this large field in physics, this is left as an open question to the real experts.

4.A Electrodynamics with differential forms

For the reader familiar with the mathematical language of differential forms, we present the electrodynamics of vortex sources for any dimension $d = D + 1$ higher than 2. For our purposes, a differential form can be thought of as something that can be integrated over; in other words: it is a density function combined with the integrand. For instance, the electric field is a 1-form $\mathsf{E} = E_i \mathrm{d}x_i = E_x \mathrm{d}x + E_y \mathrm{d}y + E_z \mathrm{d}z$. Higher forms are always obtained through the wedge product $\mathsf{a} \wedge \mathsf{b}$, which is the antisymmetrization of the tensor product of a and b. Another common operation is the Hodge dual $*\mathsf{a}$ of a, which turns an n-form into a $(d-n)$-form. For instance in three spatial dimensions

name	field	?-form	2+1d	3+1d	representative in d=3+1
electric field	E	1	1	1	$E_x\,\mathrm{d}x$
dielectric current	$\mathsf{D} = \varepsilon *_s \mathsf{E}$	d-2	1	2	$D_x\,\mathrm{d}y \wedge \mathrm{d}z$
magnetic field	B	2	2	2	$B_x\,\mathrm{d}y \wedge \mathrm{d}z$
magnetic intensity	$\mathsf{H} = \mu *_s \mathsf{B}$	d-3	0	1	$H_x\,\mathrm{d}x$
charge density	ρ	d-1	2	3	$\rho\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
current density	J	d-2	1	2	$J_x\,\mathrm{d}y \wedge \mathrm{d}z$
covariant current	$\mathsf{j} = \rho + \mathsf{J} \wedge \mathrm{d}t$	d-1	2	3	$j_x\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t$
field strength	$\mathsf{F} = \mathsf{B} + \mathsf{E} \wedge \mathrm{d}t$	2	2	2	$F_{xy}\,\mathrm{d}x \wedge \mathrm{d}y$
gauge potential	A	1	1	1	$A_x\,\mathrm{d}x$
vortex source	J^{V}	d-2	1	2	$J^{\mathrm{V}}_{xy}\,\mathrm{d}x \wedge \mathrm{d}y$
Lagrangian density	$\mathscr{L}$	d	3	4	$\mathscr{L}\,\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

Table 4.1: Electrodynamical quantities in differential forms. Here ε and μ are the electric permittivity and the magnetic permeability, and $*_s$ is the spatial Hodge dual. Other factors of c are suppressed. Minus signs are subject to convention.

$*\mathsf{E} = E_x \mathrm{d}y \wedge \mathrm{d}z + E_y \mathrm{d}z \wedge \mathrm{d}x + E_z \mathrm{d}x \wedge \mathrm{d}y$. For a pedagogical introduction to differential forms in Maxwell electrodynamics see [77].

In the familiar case of $d = 3+1$, a 1-form is a line density or "field intensity" like the electric field; a 2-form is a surface density or flux density like the magnetic field; a 3-form is a volume density like the charge density. Confusion may arise when it is not immediately clear whether an object is an n-form or a $d-n$-form, which is important for generalization to other dimensions. We distinguish the regular Hodge dual $*$ from the spatial Hodge dual $*_s$, where the latter does not involve the temporal dimension. The exterior derivative operator is $\mathsf{d} = \frac{\partial}{\partial t}\,\mathrm{d}t \wedge + \sum_i \frac{\partial}{\partial x_i}\,\mathrm{d}x_i \wedge$, and the one with only spatial components is $\mathsf{d}_s = \sum_i \frac{\partial}{\partial x_i}\,\mathrm{d}x_i \wedge$. The Leibniz rule is $\mathsf{d}(\mathsf{a} \wedge \mathsf{b}) = \mathsf{d}\mathsf{a} \wedge \mathsf{b} + (-1)^r \mathsf{a} \wedge \mathsf{d}\mathsf{b}$, where a is an r-form. This can be used for partial integration.

In table 4.1 we have listed the differential forms of the relevant fields. Some of these definitions seem perhaps unfamiliar. In particular, we are used to thinking of the magnetic field as a vector field; however, its solenoidal nature is typical of a two-form. This becomes even more clear when it is expressed as the curl of the vector potential $\mathsf{B} = \mathsf{d}_s\mathsf{A}$, which holds in $3+0$

dimensions. Also the current density J is naturally a flux or a 2-form, but its generalization is as a $d-2$-form. One way to see that this must be so, is to write down the continuity equation in differential forms,

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0 \qquad \rightarrow \qquad (\partial_t \rho + \mathsf{d}_s \mathsf{J}) \wedge \mathsf{d}t = \mathsf{dj} = 0. \tag{4.60}$$

The current density appearing as a vector field for instance in Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$ is actually the spatial Hodge dual of J.

We shall now write down the familiar expressions of Maxwell electrodynamics. The Lagrangian density is a spacetime volume density. All terms must therefore combine into d-forms. The field strength is $\mathsf{F} = \mathsf{dA}$. From this definition it is clear that the gauge transformations $\mathsf{A} \rightarrow \mathsf{A} + \mathsf{d}\xi$, with ξ any 0-form, leave the field strength unchanged, since $\mathsf{d}^2 = 0$. The field strength is contracted with its dual to obtain a d-form in the Lagrangian. The sources couple to the gauge potential (this is another reason why the source is a $d-1$ form). The Maxwell action is then,

$$S = \int -\mathsf{F} \wedge *\mathsf{F} + \mathsf{A} \wedge \mathsf{j}. \tag{4.61}$$

The second term is also invariant under the same gauge transformations, provided that $\mathsf{dj} = 0$, the continuity equation. The Euler–Lagrange equations are,

$$\mathsf{d} \frac{\partial \mathsf{L}}{\partial \mathsf{dA}} - \frac{\partial \mathsf{L}}{\partial \mathsf{A}} = 0, \tag{4.62}$$

resulting in the inhomogeneous Maxwell equations,

$$\mathsf{d} * \mathsf{dA} = \mathsf{d} * \mathsf{F} = -\mathsf{j}, \qquad (\partial_\mu F_{\mu\nu} = -J_\nu). \tag{4.63}$$

Applying the exterior derivative on this equation directly leads to the continuity equation, since $\mathsf{d}^2 = 0$. Similarly, from the definition $\mathsf{F} = \mathsf{dA}$ it immediately follows that,

$$\mathsf{dF} = 0, \tag{4.64}$$

which are the homogeneous Maxwell equations, or in this context rather the Bianchi identities.

Now let us repeat the reasoning of section 4.2. In the absence of monopole sources J, we have both $\mathsf{d} * \mathsf{F} = 0$ and $\mathsf{dF} = 0$. This implies that the field strength has become "pure gauge". The first of these equations still holds when we add any 1-form ξ as $*\mathsf{F} \rightarrow *\mathsf{F} + \mathsf{d}\xi$. The original Bianchi identities are not invariant under these transformations. The dual field strength $*\mathsf{F}$ turns into a gauge potential, and is accompanied by its own field strength $\mathsf{K} = \mathsf{d} * \mathsf{F}$, which contracts with its dual in the Lagrangian. The field strength can couple to a $d-2$-form source, which we anticipatively denote by J^{V}, provided that $\mathsf{d} * \mathsf{J}^{\mathrm{V}} = 0$. Indeed,

$$\mathsf{F} \wedge \mathsf{J}^{\mathrm{V}} \to \mathsf{F} \wedge \mathsf{J}^{\mathrm{V}} + *\mathsf{d}\xi \wedge \mathsf{J}^{\mathrm{V}} = \mathsf{F} \wedge \mathsf{J}^{\mathrm{V}} - \xi \wedge \mathsf{d} * \mathsf{J}^{\mathrm{V}} = \mathsf{F} \wedge \mathsf{J}^{\mathrm{V}}. \tag{4.65}$$

The second step is achieved by partial integration, and the last equality holds if the vortex current is conserved, $\mathsf{d} * \mathsf{J}^{\mathrm{V}} = 0$. The action for vortices directly sourcing the field tensor is, (with $\mathsf{G} = *\mathsf{F}$),

$$S = \int -\mathsf{K} \wedge *\mathsf{K} + \mathsf{F} \wedge \mathsf{J}^{\mathrm{V}} = \int -\mathsf{K} \wedge *\mathsf{K} + \mathsf{G} \wedge *\mathsf{J}^{\mathrm{V}}. \tag{4.66}$$

Variation with respect to G leads to,

$$* \mathsf{d} * \mathsf{d}\mathsf{G} = -\mathsf{J}^{\mathrm{V}}. \tag{4.67}$$

This equation corresponds to $\epsilon_{\kappa\lambda\mu\nu}\partial^2 F_{\mu\nu} = -J^{\mathrm{V}}_{\kappa\lambda}$ as in Eq. (4.43), but is valid in any dimension. The addition of a Meissner term results in

$$S = \int -\mathsf{K} \wedge *\mathsf{K} - \mathsf{G} \wedge *\mathsf{G} + \mathsf{G} \wedge *\mathsf{J}^{\mathrm{V}}. \tag{4.68}$$

and,

$$* \mathsf{d} * \mathsf{d}\mathsf{G} - \mathsf{G} = -\mathsf{J}^{\mathrm{V}}. \tag{4.69}$$

This is the equation of motion for $d-1$-dimensional superconductors, which have $d-2$-dimensional vortex world branes J^{V}.

Chapter 5

Type-II Mott insulators

In chapter 3 we have seen that the Bose-Mott insulator is in fact a disordered superfluid, where the superfluid vortices have proliferated, and furthermore that the Bose-Mott insulator supports vortices of its own, in the form of lines of supercurrent. This we coined the type-II Bose-Mott insulator. In chapter 4 we have seen how to formulate a relativistic description of Abrikosov vortices in a superconductor, and thus how to wire in electromagnetism. It is now time to combine the acquired knowledge, and to look at vortices in the charged Bose-Mott insulator.

The essence is very much the same as the charge-neutral case, but the outcome is striking: lines of electric current piercing through an otherwise insulating slab of material. These Mott vortex lines contain a quantum of electric current, just as Abrikosov vortices have a magnetic flux quantum. In fact, almost all of the electrodynamic properties of a type-II superconductor are mirrored in the type-II Mott insulator, where "magnetic field" has to be substituted for "electric current".

There are a few notable exceptions to this principle. Firstly, the electric current $J_\mu = \frac{e^*}{\hbar} w_\mu$ is a vector quantity, whereas the magnetic field or rather the Maxwell field strength is a 2-form. As such, the coupling to the vortex world sheet of the current is mathematically different when compared to the magnetic field. The reason for this is easily understood intuitively: the vortex is a line of electric current, which is electric charge in motion. If such a line moves, it is just that the microscopic charges are moving in a different direction than 'straight up'. Compare this to a magnetic field, which in motion generates an electric field. Surely this is a rather different situation.

Secondly, in a superconductor one has the true vacuum where electromagnetic fields are free, and the Meissner state where those fields are expelled. Now the Bose-Mott insulator mimics the Meissner state, yet for electric current instead of magnetic field; the superconductor where current is free mimics the vacuum; but on top of that we still have the real vacuum, and this has no counterpart in superconductivity. Therefore the physical situation is even richer than for type-II superconductors.

In this chapter we will repeat the duality calculation for charged superfluids, that is, a superfluid made out of Cooper pairs. First we will present a short exposition of the realization of such systems in actual materials. Afterwards considerable time will be spent on the nature of the Mott vortex world sheets. Then we collect the relevant physical observables from the equations of motion. All effects are collected in a phase diagram. And lastly, we present a host of possible experimental setups that may be able to identify the vortices in the Mott insulator.

5.1 Charged superfluid–insulator transitions

There are several systems whose properties are principally that of charged bosons, with either weak (superfluid) or strong (insulating) effective interactions. The very well-controlled optical lattice systems mentioned before [50] do not fall into that category as the strong repulsive interaction between charged atoms would dominate the subtle quantum statistical effects.

5.1.1 Arrays of Josephson junctions

Since the 1990s several groups devoted their time to making structures out of superconducting components. Most notable are the arrays of Josephson junctions. These are two-dimensional lattices of superconducting islands with charging energy C which are connected by weak links with Josephson coupling J. These systems are remarkably well described by the Bose-Hubbard model of §2.3, where the boson repulsion U is as the inverse charging energy $1/C$. Good reviews are Refs. [55, 56].

Since they are constructed out of superconducting materials, they are of course electrically charged. As such, they can be probed by electromagnetic means. Also, vortices in the insulating state would be of the kind described in this chapter.

All in all, this seems like an ideal system to look for type-II behaviour in the Mott insulating state, because the level of control one has in the synthesis of the arrays, and techniques that have already been developed over the past two decades. There is one big caveat however: they have always been restricted to two-dimensional systems. It turns out to be very hard to make truly three-dimensional lattices of this kind. Of course, the two-dimensional version will also have Mott vortices (vortex pancakes), but that prediction is not as striking as the real three-dimensional vortex lines.

5.1.2 Underdoped cuprate superconductors

In 1986 Bednorz and Müller discovered superconductivity in an otherwise very poorly conducting ceramic copper-oxide material up to an unprecedented high temperature. This sparked a true frenzy of research chasing experimentally after new materials with ever higher T_c's and theoretically after the underlying physical mechanism. Up to now, the first endeavour has progressed reasonably well, while the latter has been stuck for a long time. However, these days most scientists in the field would agree that the unconventional properties of the cuprate (and other high-T_c) superconductors lie more in the 'normal' state than in the superconducting one.

The critical temperature T_c below which superconductivity prevails is a function of the chemical doping (adding electron or hole carriers) of the material. The highest T_c is said to be at *optimal doping* (OP). With fewer carriers it is *underdoped* (UD), with more it is *overdoped* (OD). On the overdoped side, the normal state above T_c is much like a regular Fermi liquid (normal metal). But the properties on the underdoped side of the cuprates like $La_{2-x}Sr_xCuO_4$ or $YBa_2Cu_3O_{7-\delta}$ are very peculiar indeed. People find all kinds of electronic ordering [78] like stripes [79, 80], orbital currents [81] and recently also quantum nematics [82–84]. Furthermore a second energy gap (distinct from the superconducting gap) shows up the single-electron spectrum, dubbed the *pseudogap*. See the phase diagram in figure 5.1.

A hypothesis that has many proponents is that in the pseudogap region, electrons do already combine into *preformed* Cooper pairs, which causes the energy gap by the removal of electron states, but the phase fluctuations are too strong to induce long-range phase coherence, such that there is no superconducting order yet [85, 86]. Viewed from the opposite side starting from

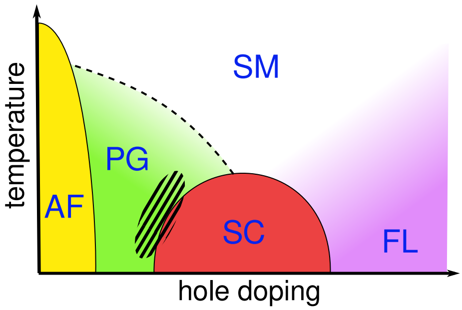


Figure 5.1: Sketch of the generic phase diagram of hole-doped cuprate superconductors. The only undisputed phases are the antiferromagnetic Mott insulator (AM, yellow), superconductor (SC, red) and Fermi liquid (FL, purple). Right above the superconducting dome is a region with electric resistivity that grows linearly with temperature, and is therefore often referred to as strange metal (SM, white). In green is shown the pseudogap region (PG), with the appearance of an additional gap in the single electron response. In is unclear whether there is a phase transition or a crossover to the strange metal. The hatched area crudely indicates where interesting electronic ordering is found, and also for instance a large Nernst effect; this is also the first candidate to look for type-II Mott insulators.

the superconductor: first the phase coherence is destroyed accompanied by the loss of superconductivity, and only at a higher temperature do the Cooper pairs break up. If true, this implies that there is a region in the phase diagram with phase-disordered Cooper pairs, c.q. charged bosons. Therefore this state would actually be a charged Bose-Mott insulator, the topic of this chapter.

This is beneficial in two ways: firstly, this is a suitable testing ground to go and find the type-II Mott insulator and the Mott vortices. These materials have been very well studied, and there are many techniques for both synthesis and experimental characterization. Conversely, if the type-II Mott behaviour were to been found, it would constitute strong evidence for the pseudogap regime as a phase-disordered superconductor.

5.2 Vortex world sheets coupling to supercurrent

In this section we will use physical arguments to determine the correct form of the minimal coupling of the Mott vortices to the dual gauge field and therefore the supercurrent. The only ingredient that we need on top of the discussion in §3.4 is that the supercurrent is now electrically charged, with the correspondence $J_\mu = \frac{e^*}{\hbar} w_\mu$. The full calculation will be performed in the next section; here we only want to illustrate to the reader how to view relativistically the current-carrying vortex, in contrast to the Abrikosov vortices of §4.2.

5.2.1 Limiting to 3+0 and 2+1 dimensions

To obtain the appropriate formulation in the fully relativistic 3+1 dimensional case, it will prove very useful to understand first the special cases of 3+0 and 2+1 dimensions, to both of which the full model must reduce as a lower-dimensional hyperspace cut of the 3+1 dimensional spacetime.

In 3+0 dimensions, the minimal coupling of the dual gauge field b_k, which is now a vector field, to the disorder parameter Φ is straightforward,

$$\mathscr{L}_{\text{min.coup.}} = |(\partial_k - \mathrm{i} b_k)\Phi|^2 = |\Phi|^2(\partial_k \phi - b_k)^2. \tag{5.1}$$

In the equations of motion, we then find,

$$\partial_k \phi - b_k = 0, \tag{5.2}$$

and acting on this expression with $\epsilon_{mnk}\partial_n$ leads to,

$$w_m = \epsilon_{mnk}\partial_n b_k = \epsilon_{mnk}\partial_n\partial_k\phi = \mathscr{J}^{\mathrm{V}}_m, \tag{5.3}$$

where the last equality is the definition of the vortex current $\mathscr{J}^{\mathrm{V}}_m$. This expression agrees with the intuition that a vortex line in a Mott insulator is parallel to the electric current $J^{\mathrm{EM}}_m = \frac{e^*}{\hbar} w_m$.

As we mentioned before, the minimal coupling Eq. (3.34),

$$\mathscr{L}_{\text{min.coup.}} = \frac{1}{2}|(\partial_\mu - \mathrm{i}\epsilon_{\mu\|\kappa\lambda} b_{\kappa\lambda})\Phi|^2 \tag{5.4}$$

does not specialize back to back to Eq. (5.1) in 3+0 dimensions.

We need to find another form for the minimal coupling, that satisfies the following conditions,

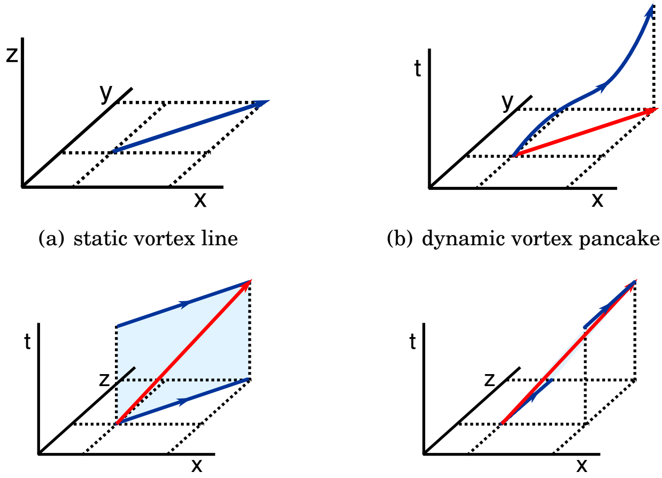
1. The term in the Lagrangian is equivalent to Eq. (3.23), such that only a single additional degree of freedom arises in the Higgs phase;
2. The equations of motion reduce naturally to the cases of 3+0 and 2+1 dimensions.

The problem of matching the two-form gauge field $b_{\kappa\lambda}$ to the one-form condensate phase mode $\partial_\mu\phi$ is equivalent to matching the two-form vortex world sheet $\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}$ to the one-form supercurrent w_μ. Fortunately, we can fall back to the limiting cases of 2+1 and 3+0 dimensions, representing a dynamic vortex pancake and a static vortex line respectively.

5.2.2 Static vs. dynamic vortex lines

In 3+0 dimensions a vortex line $\mathscr{J}^{\mathrm{V}}_l$ in the Mott insulator is just a static line of electric current $J^{\mathrm{EM}}_l \sim w_l$. Since here the time dimension is absent, the three components of the vortex line correspond to the temporal (density) components of the vortex world sheet $\mathscr{J}^{\mathrm{V}}_{tl}$. Therefore these temporal components of world sheet surface elements correspond to the spatial current $\mathscr{J}^{\mathrm{V}}_{tl} \sim w_l$.

In 2+1 dimensions we have a vortex pancake in the spatial xy-plane, which is therefore represented by a scalar quantity, the charge density w_t. When this vortex pancake moves, its charged vortex core moves, which is equivalent to having an electric current as witnessed by the continuity equation $\partial_t w_t + \partial_k w_k = 0$. Since the vortex pancake can be viewed as a slice through



(a) static vortex line (b) dynamic vortex pancake

(c) static vortex sheet (d) dynamic vortex sheet

Figure 5.2: (a) Static vortex line in the xy-plane; the current flows through the line. (b) Vortex pancake moving in time (blue). The associated current in the spatial direction is shown in red. (c) Static vortex line in the xz-plane moving straight up in time. (d) A vortex line in the z-direction moving in the x-direction through time. The last two world sheet configurations correspond to the same electromagnetic current (red).

4-dimensional spacetime orthogonal to the third spatial direction l, this suggests that $\mathscr{J}^{\mathrm{V}}_{\kappa l} = \frac{e^*}{\hbar} w_\kappa$.

So here we find electric current as well, but of a different origin: in 3+0 we have a static line *through which* the current is flowing, whereas in 2+1 dimensions the *motion of the vortex itself* causes electric current. Therefore in 3+1 dimensions, we must have both of these contributions.

This is depicted in Figure 5.2. The static vortex line in the xz-plane that moves straight up in time generates the same electric current as a vortex line that is always along the z-direction but moves in the x-direction through time. In other words: the current in the z-direction can originate from the density of vorticity in the z-direction $\mathscr{J}^{\mathrm{V}}_{tz}$; or from lines along x or y that move in the z-direction, represented by $\mathscr{J}^{\mathrm{V}}_{az}$, $a = x, y$. The total current in the z-direction therefore is,

$$w_z \sim \mathscr{J}^{\mathrm{V}}_{tz} + \mathscr{J}^{\mathrm{V}}_{xz} + \mathscr{J}^{\mathrm{V}}_{yz} = \sum_\kappa \mathscr{J}^{\mathrm{V}}_{\kappa z}. \tag{5.5}$$

Now for the charge density[1] w_t, we note that is is an undirected quantity. The charge density does not care in which direction the vortex line is pointing. Therefore the charge density gets contributions from world sheet elements that represent the density of vorticity in all spatial directions, $w_t \sim \sum_\kappa \mathscr{J}^{\mathrm{V}}_{\kappa t}$. Therefore we may conclude that,

$$w_\lambda \sim \sum_\kappa \mathscr{J}^{\mathrm{V}}_{\kappa\lambda}. \tag{5.6}$$

The continuity equation for the electric current $\partial_\lambda w_\lambda = 0$ is satisfied due to the no-monopoles condition of the vortex world sheet $\partial_\lambda \mathscr{J}^{\mathrm{V}}_{\kappa\lambda} = 0$. In the limiting cases of 3+0 or 2+1 dimensions, for each component of the current w_λ there is only a single contribution from the vortex (world) line, and then there is no summation. The 3+1 dimensional vortex world sheet $\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}$ reduces to the special limits of 2+1 and 3+0 dimension as follows. The static vortex line in 3+0 dimensions has only the density components, or $\mathscr{J}^{\mathrm{V}}_l = \mathscr{J}^{\mathrm{V}}_{tl}$. For 2+1 dimensions, we picture a vortex line in the z-direction, and we take a slice in the txy-hyperplane; then $\mathscr{J}^{\mathrm{V}}_\kappa = \mathscr{J}^{\mathrm{V}}_{\kappa z}$.

5.2.3 Minimal coupling by sum over vortex components

We propose the following minimal coupling prescription, that satisfies the above mentioned conditions and results in Eq. (5.6),

$$\mathscr{L}_{\text{min.coup.}} = |(\frac{1}{2}\sum_\alpha \delta_{\alpha\kappa}\partial_\lambda - \mathrm{i}b_{\kappa\lambda})\Phi|^2 = |\Phi|^2(\frac{1}{2}\sum_\alpha \delta_{\alpha\kappa}\partial_\lambda\phi - b_{\kappa\lambda})^2. \tag{5.7}$$

This is the form already encountered in Eq. (3.30), and we have now presented the physical reason for this form. If we expand the square, we find,

$$\begin{aligned}
(\frac{1}{2}\sum_\alpha \delta_{\alpha\kappa}\partial_\lambda\phi - b_{\kappa\lambda})^2 &= \frac{1}{4}\sum_\alpha \delta_{\alpha\kappa}\partial_\lambda\phi \sum_\beta \delta_{\beta\kappa}\partial_\lambda\phi - b_{\kappa\lambda}\sum_\alpha \delta_{\alpha\kappa}\partial_\lambda\phi + b^2_{\kappa\lambda} \\
&= (\frac{1}{4}\sum_{\alpha\beta}\delta_{\alpha\beta})(\partial_\lambda\phi)^2 + \sum_\alpha \phi\partial_\lambda b_{\alpha\lambda} + b^2_{\kappa\lambda} \\
&= (\partial_\lambda\phi)^2 + b^2_{\kappa\lambda} \qquad \text{(Lorenz gauge).}
\end{aligned} \tag{5.8}$$

[1]Even though the Mott insulator as a whole is electrically neutral, the vortex lines carry current because the Cooper pairs can move freely. Therefore this charge density is just the density of Cooper pairs, which is clearly quantized in units of $e^* = 2e$, and the balancing positive charge is not taken into consideration. The same applies of course in a current-carrying metal wire.

In the second step we have performed partial integration, and in the last step we have enforced the Lorenz gauge condition $\partial_\kappa b_{\kappa\lambda} = 0$. Here we see that this form is indeed equal to that of Eq. (3.23), where $\partial_\lambda \phi$ represents the longitudinal component of w_μ and the three degrees of freedom of $b_{\kappa\lambda}$ remaining after the gauge fix are the transversal ones.

Next, in the equations of motion, we will encounter the term,

$$\frac{\partial \mathscr{L}}{\partial b_{\kappa\lambda}} = \frac{1}{2}\sum_\alpha \delta_{\kappa\alpha}\partial_\lambda\phi - \frac{1}{2}\sum_\alpha \delta_{\lambda\alpha}\partial_\kappa\phi - b_{\kappa\lambda}. \tag{5.9}$$

Acting on this expression with $\epsilon_{\mu\nu\kappa\lambda}\partial_\nu$ leads to,

$$\begin{aligned} \frac{1}{2}\sum_\kappa \epsilon_{\mu\nu\kappa\lambda}\partial_\nu\partial_\lambda\phi - \frac{1}{2}\sum_\lambda \epsilon_{\mu\nu\kappa\lambda}\partial_\nu\partial_\kappa\phi - \epsilon_{\mu\nu\kappa\lambda}\partial_\nu b_{\kappa\lambda} &= \sum_\kappa \epsilon_{\kappa\mu\nu\lambda}\partial_\nu\partial_\lambda\phi - w_\mu \\ &= \sum_\kappa \mathscr{J}^{\mathrm{V}}_{\kappa\mu} - w_\mu. \end{aligned} \tag{5.10}$$

This precisely agrees with Eq. (5.6).

There are three details that may raise some concern. Firstly, the expression in Eq. (5.7) is not antisymmetric under the interchange $\kappa \leftrightarrow \lambda$. We could write down a fully antisymmetric form, but that would leads to contractions $\sim \sum_\lambda \partial_\lambda \phi$. We suspect that such terms would fall within the gauge volume or would otherwise be dynamically constrained. But in fact, nothing requires the term to be antisymmetric in the first place. In the relevant quantities, such as the vortex current $\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}$, the antisymmetry follows automatically. The expression in Eq. (5.9) is one example of this.

The next point is that the expression in Eq. (5.7) is not strictly gauge invariant. The gauge transformations for the two-form dual gauge field are Eq. (3.5). The resolution of the alternative form Eq. (3.26) was to explicitly leave the gauge volume out of the minimal coupling. But this expression Eq. (5.7) is to be taken gauge fixed. This is not an actual problem, as the physical field content is dictated by the currents, as in Eq. (3.23). As of yet, we have not found a way to balance the three gauge degrees of freedom of the two-form gauge field with the condensate phase mode. It remains our conviction that the minimal coupling to a vector field is rather special in this regard.

Lastly, as mentioned in §3.4.5, there is as of yet no way to complete the "duality squared" procedure with this form of the minimal coupling. Since we know that the outcome will be fine using the alternate form we leave this aside, and focus here on the more interesting vortices in the Mott insulator themselves.

5.3 Charged vortex duality

Here we perform the duality transformation of §2.4.7 for 3+1 dimensions. About half of the calculation was already done in §4.3, but we now find it convenient here to work in imaginary time.

5.3.1 Dual superconductor

Then starting with the dimensionless action of the Ginzburg–Landau superconductor Eq. (2.37),

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \; -\frac{1}{2g}(\partial^{\mathrm{ph}}_\mu \varphi - A^{\mathrm{ph}}_\mu)^2 - \frac{1}{4\mu} F^2_{\mu\nu}, \tag{5.11}$$

we will end up with the Euclidean version of Eq. (4.36),

$$Z = \int \mathscr{D} J^{\mathrm{V}}_{\kappa\lambda} \mathscr{D} A_\mu \mathscr{F}(A_\mu) \mathscr{D} b_{\kappa\lambda} \mathscr{F}(b_{\kappa\lambda}) \mathrm{e}^{-\int \mathscr{L}_{\mathrm{dual}}}, \tag{5.12}$$

$$\mathscr{L}_{\mathrm{dual}} = \frac{1}{2} g (\epsilon_{\mu\nu\kappa\lambda} \partial^{\mathrm{ph}}_\nu b_{\kappa\lambda})^2 - b_{\kappa\lambda} J^{\mathrm{V}}_{\kappa\lambda} + \epsilon_{\mu\nu\kappa\lambda} \partial^{\mathrm{ph}}_\nu b_{\kappa\lambda} A^{\mathrm{ph}}_\mu - \frac{1}{4\mu} F^2_{\mu\nu}. \tag{5.13}$$

Here the coupling constants are,

$$\frac{1}{g} = \frac{Ja}{\hbar c_{\mathrm{ph}}}, \qquad \frac{1}{\mu} = \frac{\hbar a^{D-3}}{\mu_0 c_{\mathrm{ph}} e^{*2}}. \tag{5.14}$$

The first is always dimensionless, the last is dimensionless if $D = 3$, which is the case we are interested in, and we specialize to 3+1 dimensions from now on.

In these dimensionless units, the charge of the vortex minimal coupling is 1, which was the reason for rescaling to these units in the first place. The action above describes one or several individual (Abrikosov) vortex sources that interact via the mediation of the dual gauge fields $b_{\kappa\lambda}$. These gauge fields are the duality transforms of the original Goldstone modes φ. They remember that the bosons are electrically charged by also coupling to the electromagnetic field A_μ. If one were to integrate out the dual gauge fields, one would find an action of charged vortices that couple to each other non-locally. They would have long-range interactions were it not for the electromagnetic fields, which induce Meissner screening.

5.3.2 Vortex proliferation

This is however not what we are interested in at the moment. We are going to proceed and let the vortex strings proliferate into the 'string foam' as explained in §3.2. The disorder parameter Φ is the 'density of the string foam', and the minimal coupling to the gauge field is dictated by the considerations of §5.2. Thus we find,

$$\begin{aligned}\mathscr{L} = \frac{1}{2}g(\epsilon_{\mu\nu\kappa\lambda}\partial^{\text{ph}}_{\nu}b_{\kappa\lambda})^2 + \epsilon_{\mu\nu\kappa\lambda}\partial^{\text{ph}}_{\nu}b_{\kappa\lambda}A^{\text{ph}}_{\mu} - \frac{1}{4\mu}F^2_{\mu\nu} \\ + \frac{1}{2}|(\frac{1}{2}\sum_{\alpha}\delta_{\alpha\kappa}\partial_{\lambda} - \mathrm{i}b_{\kappa\lambda})\Phi|^2 + \frac{\tilde{\alpha}}{2}|\Phi|^2 + \frac{\tilde{\beta}}{4}|\Phi|^4.\end{aligned} \tag{5.15}$$

Here we have added Ginzburg–Landau potential energy terms for the dual order parameter, which we will neglect from now on. If $\tilde{\alpha} < 0$, the dual order parameter obtains an expectation value $\langle\Phi\rangle = \sqrt{\frac{|\tilde{\alpha}|}{\tilde{\beta}}} \equiv \Phi_{\infty}$. This signals the phase transition to the Bose-Mott insulator, with the Mott gap represented by $|\Phi|^2$.

What we would like to do, similar to the procedure in §3.4.5, is dualize the dual phase field ϕ to a conserved current v_{μ}, integrate out the smooth part, define the Mott vortex current $\mathscr{J}^{\text{V}}_{\kappa\lambda} = \epsilon_{\kappa\lambda\nu\mu}\partial_{\nu}\partial_{\mu}\phi$ and integrate out the current v_{μ} to find the direct coupling of the Mott vortex current to the supercurrent gauge field $b_{\kappa\lambda}$. However, as mentioned before, I have not been able to find a consistent way of doing it for this form of the minimal coupling. Fortunately, the action (5.15) is sufficient to find the Mott vortex electrodynamics, just as it was for the Abrikosov vortices in chapter 4.

5.4 Phenomenology of Mott vortices

In this section we derive observable quantities of the Bose-Mott insulator and its vortices. This mostly follows the same reasoning as for the regular Ginzburg–Landau model of §2.1, see also e.g. [51, ch.4].

5.4.1 Equations of motion

We calculate the equations of motions by varying Eq. (5.15) with respect to $\bar{\Phi}$, $b_{\kappa\lambda}$ and A_μ.

$$(\frac{1}{2}\sum_\alpha \delta_{\kappa\alpha}\partial^{\text{ph}}_\lambda - \mathrm{i}b_{\kappa\lambda})^2\Phi - \tilde{\alpha}\Phi - \tilde{\beta}|\Phi|^2\Phi = 0, \tag{5.16}$$

$$-g\epsilon_{\kappa\lambda\nu\mu}\partial^{\text{ph}}_\nu w_\mu + |\Phi|^2\Big(\frac{1}{2}\sum_\alpha(\delta_{\kappa\alpha}\partial^{\text{ph}}_\lambda\phi - \delta_{\lambda\alpha}\partial^{\text{ph}}_\kappa\phi) - b_{\kappa\lambda}\Big) = \frac{1}{2}\epsilon_{\kappa\lambda\mu\nu}F^{\text{ph}}_{\mu\nu}, \tag{5.17}$$

$$\frac{1}{\mu}\partial_\mu F_{\mu\nu} = -w^{\text{ph}}_\nu. \tag{5.18}$$

Here we have substituted definitions of $w_\mu = \epsilon_{\mu\nu\kappa\lambda}\partial^{\text{ph}}_\nu b_{\kappa\lambda}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The superscripts on $F^{\text{ph}}_{\mu\nu}$ and w^{ph}_μ indicate that those quantities carry a velocity ratio in the temporal components: $F^{\text{ph}}_{tn} = \frac{c}{c_{\text{ph}}}F_{tn}$ and $w^{\text{ph}}_t = \frac{c}{c_{\text{ph}}}w_t$. The dimensionful versions of these equations are,

$$-a^2(\sum_\alpha \delta_{\kappa\alpha}\partial^{\text{ph}}_\lambda - \mathrm{i}\frac{a}{\hbar c_{\text{ph}}}b_{\kappa\lambda})^2\Phi + \tilde{\alpha}\Phi + \tilde{\beta}|\Phi|^2\Phi = 0, \tag{5.19}$$

$$-ga^2\epsilon_{\kappa\lambda\nu\mu}\partial^{\text{ph}}_\nu w_\mu + |\Phi|^2\Big(\frac{1}{2}\sum_\alpha\frac{\hbar c_{\text{ph}}}{a}(\delta_{\kappa\alpha}\partial^{\text{ph}}_\lambda\phi - \delta_{\lambda\alpha}\partial^{\text{ph}}_\kappa\phi) - b_{\kappa\lambda}\Big) = \frac{1}{2}c_{\text{ph}}e^*\epsilon_{\kappa\lambda\mu\nu}F^{\text{ph}}_{\mu\nu}, \tag{5.20}$$

$$\frac{1}{\mu_0}\partial_\mu F_{\mu\nu} = -\frac{e*}{\hbar}w^{\text{ph}}_\nu = -J^{\text{ph}}_{\text{s}\;\nu}. \tag{5.21}$$

In the last equality we used the definition of the supercurrent $J^{\text{s}}_\nu = \frac{e^*}{\hbar}w_\nu$. Note that the last two equations reduce to the equations of motion for the superconductor in the limit $|\Phi|^2 \to 0$. The last equation is the same with or without the Mott condensate, and just reflects the generation of an electromagnetic field by a current. The second equation is basically the extension of the Meissner screening of the electric current as in Eq. (4.54), but is now sourced by Mott vortices ϕ_{MV}. We are now set to discuss the physical content of these equations.

5.4.2 Maxwell equations

The last equation Eq. (5.21) is clearly the inhomogeneous Maxwell equations for a source term $J^{\text{ph}}_{\text{s}\;\nu}$. This equation carries over from the superconductor, and does not pertain as such to the Mott insulating state. The insulating behaviour is due to the screening of the electric current, which is represented by the term $\sim |\Phi|^2$. Therefore, Eq. (5.21) is just the vacuum contribution to electric and magnetic fields generated by a current source.

5.4.3 Penetration depth

The dual penetration depth $\tilde{\lambda}$ sets the length scale up to which an electric current penetrates in the Mott insulating region. To find it we act on Eq. (5.20) with $\epsilon_{\rho\sigma\kappa\lambda}\partial_\sigma^{\mathrm{ph}}$. Contracting repeated indices, and using $\partial_\rho^{\mathrm{ph}} w_\rho = 0$, we find in the dual London limit $|\Phi| = \Phi_\infty$,

$$ga^2(\partial_\mu^{\mathrm{ph}})^2 w_\rho - \Phi_\infty^2 w_\rho + c_{\mathrm{ph}} e^* \partial_\mu^{\mathrm{ph}} F_{\mu\rho}^{\mathrm{ph}} = -\Phi_\infty^2 \frac{\hbar c_{\mathrm{ph}}}{a} \sum_\kappa \mathscr{J}_{\kappa\rho}^{\mathrm{V}}. \tag{5.22}$$

Here we used the definition of the vortex current Eq. (3.33). The interpretation of this equation is as follows: a supercurrent w_ρ can be generated by a vortex source $\mathscr{J}_{\kappa\rho}^{\mathrm{V}}$. This current is "dual Meissner screened" by the Mott condensate Φ_∞ as witnessed by the second term; but there is also some electromagnetic screening from the 'backreaction' of the induced electromagnetic field. In order to see this, we would like to substitute Eq. (5.21) in this equation. This is however complicated by the additional factors of $\frac{c}{c_{\mathrm{ph}}}$, which will clutter up the full expression. Recall however that this electromagnetic screening originates from the superconductor, and must comply with Eq. (4.54). Thus let us take the simplest case, that of static limit with only stationary flow: all time derivatives set to zero. Then we can use $\partial_m F_{mn} = -\frac{\mu_0 e^*}{\hbar} w_n$, to find in the absence of vortex sources,

$$\begin{aligned} ga^2 \nabla^2 w_n - \Phi_\infty^2 w_n - \frac{\mu_0 {e^*}^2 c_{\mathrm{ph}}}{\hbar} w_n = 0, &\qquad \text{or} \\ \nabla^2 w_n - \frac{\hbar \rho_{\mathrm{s}}}{c_{\mathrm{ph}} m^*} \Phi_\infty^2 w_n - \frac{1}{\lambda^2} w_n = 0 & \end{aligned} \tag{5.23}$$

Here we substituted $ga^2 = m^* c_{\mathrm{ph}}/\hbar\rho_{\mathrm{s}}$ (see §2.3.6), and used the definition of the London penetration depth $\lambda^2 = \mu_0 {e^*}^2 \rho_{\mathrm{s}}/m^*$. So we indeed find two contributions to screening of electric current. The first $\sim \Phi_\infty^2$ is due to the Mott insulator, and the second remembers that the system originated from a superconductor. This is actually rather odd: the Meissner screening is due to the fact that the superconductor wants to expel the magnetic field, which is not true for the Mott insulator. However, let us make a crude estimate of the relative strengths of the screening, by inserting the numerical values,

$$\mu_0 = 4\pi.10^{-7} \approx 10^{-6}\text{N/A}^2, \quad e^* \approx 10^{-19}\text{C}, \quad \hbar \approx 10^{-34}\text{Js}, \quad c_{\text{ph}} \approx \frac{1}{300}c \approx 10^6\text{m/s}, \tag{5.24}$$

we find that the relative strengths are

$$\frac{\text{Mott}}{\text{Meissner}} \approx \frac{\Phi_\infty^2}{\mu_0 e^{*2} c_{\text{ph}}/\hbar} \approx \frac{\Phi_\infty^2}{10^{-6}.10^{-38}.10^6.10^{34}} \approx 10^4 \Phi_\infty^2. \tag{5.25}$$

Now Φ_∞^2 is dimensionless, but as the order parameter of the Mott condensate it should be surely greater than 1. Therefore the expulsion of electric current due to the Mott term is several orders of magnitude stronger than the Meissner screening, and for all purposes the latter may be ignored, also eliminating our interpretative problem.

Hence the dual penetration depth of electric current in the Mott condensate is $\tilde{\lambda} = \sqrt{\frac{\hbar}{c_{\text{ph}} m^*} \rho_{\text{s}} \Phi_\infty^2}$. It depends on many material parameters. Here, as we often do, we encounter the combination $\rho_s \Phi_\infty^2$, which is the product of the superconducting order parameter and the Mott order parameter. At first, one may think that they should be mutually exclusive, as one has either superconducting order *or* Mott insulating order. However one must realize that the Mott insulator is made out of repelling Cooper pairs: the larger the number of Cooper pairs, as denoted by the superfluid density ρ_{s}, the stronger the electromagnetic effects such as screening. It is just Φ_∞^2 that signals the existence of the Mott state, whereas the combination $\rho_{\text{s}} \Phi_\infty^2$ is the appropriate Higgs mass.

5.4.4 Coherence length

If in Eq. (5.19) we rescale the dual order parameter Φ by extracting it by its equilibrium value $\Phi_\infty = \sqrt{\frac{|\tilde{\alpha}|}{\tilde{\beta}}}$, so $\Phi = \Phi_\infty \Phi'$, and set $b_{\kappa\lambda}$ to zero which is true deep within the Mott insulator, the equation reduces to,

$$\frac{a^2}{|\tilde{\alpha}|} (\partial_\mu^{\text{ph}})^2 \Phi' + \Phi' - \Phi'^3 = 0. \tag{5.26}$$

Hence we can define the dual coherence length $\tilde{\xi} = \frac{a}{\sqrt{|\tilde{\alpha}|}}$, which depends on the details of the dual symmetry breaking through the precise value of the Ginzburg–Landau parameter $|\tilde{\alpha}|$.

The coherence length is rather unimportant in this story. We are primarily interested in the type-II regime where vortices can arise, and then $\tilde{\xi}$ is very short, perhaps even near the lattice constant. All the questions we ask of the system are related to longer length scales. In other words, we assume the dual London limit where $|\Phi| = \Phi_\infty$ is constant, and $\tilde{\xi}$ denotes the typical scale over which variations of $|\Phi|$ are important.

5.4.5 Current quantization

Now we come to the most striking prediction: the existence of 'quantized' vortex lines of electric current. The equation (5.20) is just as the regular Ginzburg–Landau equation Eq. (2.5), and we can imagine a closed contour over which the change of the phase ϕ is a multiple of 2π, that is,

$$\oint_{\partial\mathscr{S}} \mathrm{d}x^\mu \partial_\mu \phi = 2\pi N. \tag{5.27}$$

We are free to choose this contour deep within the Mott insulator far away from the vortex line, such that the electric current in suppressed $w_\mu = 0$. Now assume there is no external electromagnetic field $F^{\mathrm{ext}}_{\mu\nu} = 0$, and the induced field is very small as argued in Eq. (5.25). Then Eq. (5.20) reduces to,

$$\frac{1}{2}\sum_\alpha \frac{\hbar c_{\mathrm{ph}}}{a}(\delta_{\kappa\alpha}\partial_\lambda\phi - \delta_{\lambda\alpha}\partial_\kappa\phi) = b_{\kappa\lambda}. \tag{5.28}$$

We restrict our attention to the case $(\kappa\lambda) = tl$, and take the static limit in which all time derivatives are set to zero. Thus we only look at a stationary current flowing through a static vortex line. Then,

$$\frac{\hbar c_{\mathrm{ph}}}{2a}\partial_l\phi = b_{tl}. \tag{5.29}$$

We take the line integral of this equation as in (5.27). On the right-hand side we invoke Stokes' theorem (cf. §2.1.2) to find,

$$\frac{\hbar c_{\mathrm{ph}}}{2a}2\pi N = \frac{\hbar c_{\mathrm{ph}}}{2a}\oint_{\partial\mathscr{S}} \mathrm{d}x^l\ \partial_l\phi = \oint_{\partial\mathscr{S}} \mathrm{d}x^l\ b_{tl} = \int_{\mathscr{S}} \mathrm{d}S_m\ \epsilon_{mnl}\partial_n b_{tl} = \int_{\mathscr{S}} \mathrm{d}S_m\ w_m. \tag{5.30}$$

In the last step we have used the definition of the dual gauge field Eq. (3.4) in the static limit. The right-hand side is the flux of current w_m through the surface $\mathscr{S}$. Since the current is expelled from the Mott insulator, this current flows through the vortex line. For the electric current I which is the flux of the current density $J_m = \frac{e^*}{\hbar} w_m$, this implies the quantization condition,

$$I_0 = \frac{e^*}{\hbar} \frac{\hbar c_{\text{ph}}}{2a} 2\pi N = \frac{1}{\Phi_0} \sqrt{UJ} 2\pi^2 N. \tag{5.31}$$

Here $\Phi_0 = h/e^*$ is the (magnetic) flux quantum and we have substituted the microscopic parameters $\sqrt{UJ} = \hbar c_{\text{ph}}/a$ from §2.3.3.

Admittedly, this is no 'true' quantization as the current quantum depends on material parameters. This is however not unexpected, since, contrary to for instance conductivity or magnetic flux, there is no combination of natural constants that results in a unit of electric current. In any case, for a certain material under fixed environmental conditions, the current should penetrate through the Mott insulator in incremental steps of size of the current quantum. From a duality perspective, it is nice that the current quantum is proportional to the inverse of the flux quantum.

If the phase velocity c_{ph} is the same or similar for the Bose-Mott insulator as for the superconductor, then we can make a quick estimate for the $N = 1$ quantum by inserting $c_{\text{ph}} \approx 10^6$m/s and $a = 10^{-10}$m, such that

$$I_0 = \frac{e^* c_{\text{ph}}}{2a} 2\pi \approx 5.10^{-3}\text{A}, \tag{5.32}$$

which seems rather large at first sight.

5.5 The phase diagram of the type-II Bose-Mott insulator

We shall now collect all acquired knowledge about the type-II Bose-Mott insulator in a phase diagram, figure 5.3. The phase is a function of three, or rather four external parameters. The quantum phase transition from a superconductor to a Bose-Mott insulator is dependent on the coupling constant $g \sim U/J$ (see §§2.3.3, 2.3.7). Next to quantum fluctuations there are thermal fluctuations at any finite temperature T. The phase diagram is presented as is common in the literature of quantum phase transitions: increasing quantum fluctuations on the horizontal axis, and temperature on the vertical axis.

On top of this we can disturb the system by external electromagnetic means. For the superconductor we know that applied magnetic field competes with the superconducting order. And in this chapter we have learned

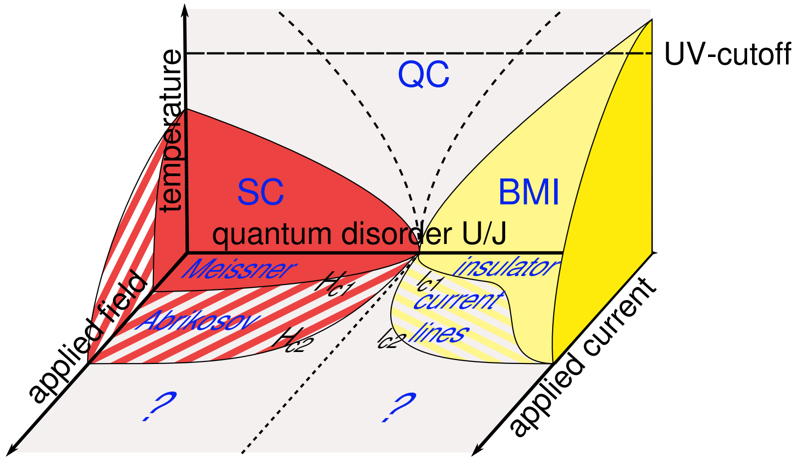


Figure 5.3: Proposed phase diagram of the type-II Bose-Mott insulators. On the horizontal axis is the strength of the quantum fluctuations that disorder the superconductor (SC) into a Bose-Mott insulator (BMI). On the vertical axis is the temperature.

In the plane there is increasing applied magnetic field H for the superconductor, resp. applied electric current I for the Bose-Mott insulator. For both the superconductor and the Bose-Mott insulator at low applied field or current, all of it is expelled by the (dual) Meissner effect. When the first flux or current quantum is generated above the lower critical field H_{c1} or current I_{c1}, the system enters in to a mixed, Abrikosov state. When the applied field or current exceeds the upper critical field H_{c2} or current I_{c2}, all of the superconductivity or insulation order is destroyed. It is unclear what will be the resulting phase at zero temperature (see text).

At finite temperature, we expect the canonical behaviour of quantum phase transitions, with a quantum critical (QC) region right above the quantum critical point. At high temperatures, the superconducting state goes over into the normal state. The Bose-Mott insulator can only originate from a Bose system of Cooper pairs; breaking up the bosons should also lead back to the normal state. When the interactions between the bosons becomes infinitely strong $U \to \infty$, the system will stay insulating. This sets a UV-limit on the applicability of our model.

that the equivalent effects in type-II Mott insulators are due to applied electric current. These two variables are drawn in the plane of the phase diagram, magnetic field for the superconducting side, and electric current for the insulating side.

There a lot going on here, so let us explore the diagram step by step. We will go through to the overly well-known superconductor in some detail, because the same reasonings will be mirrored on the insulating side.

5.5.1 Superconducting side

Surely, the superconductor holds no surprises at all. It should completely reproduce the familiar H–T-diagram found in any textbook. That is, the superconducting order persists below the critical temperature T_c, which is a decreasing function of magnetic field. When, for a particular temperature, the applied field exceeds the so-called *critical field* H_c, superconducting order is completely destroyed, and we end up in the normal state (a metal for conventional superconductors).

In a type-II superconductor, we distinguish the Meissner state below the *lower critical field* H_{c1}, and the Abrikosov state between H_{c1} and the *upper critical field* H_{c2}. The Meissner state is just as for type-I superconductors: a countercurrent will perfectly oppose the applied magnetic field. Above H_{c1}, it is energetically favourable to let magnetic field penetrate through an Abrikosov vortex line. Increasing field will create more and more of these vortices in a triangular lattice. When the applied field is so large that the vortices start to overlap (when they are approximately spaced by the penetration depth λ), superconductivity is destroyed.

In BCS theory, the superconducting gap decreases with temperature until it vanishes at T_c. The gap is proportional to the superfluid density, i.e. the 'strength' of the superconducting condensate. Therefore it is natural that the critical fields H_{c1} and H_{c2} are lower at higher temperatures, since it is easier to perturb the superconducting order.

Similarly, quantum fluctuations can diminish the superconducting order. This whole work is centred around the idea that increasing quantum disorder is just the growth of spontaneous creation and annihilation of vortex–anti-vortex pairs. Therefore increasing quantum fluctuations has the same effect as increasing thermal fluctuations: it is easier to destroy the superconducting condensate, so that the critical applied fields are lower. The situation for zero temperature and high applied field will be discussed at the end of this section.

5.5.2 Insulating side

The Bose-Mott insulator basically mimics the superconductor, where applied current takes the role of applied magnetic field. Some exceptions are foreseen on simple physical grounds as we proceed.

The point of departure is the no-fluctuations, no-applied current regime, where the system is just a “boring” Bose-Mott insulator. Approaching the quantum phase transition $U/J \to 1$, the bosons repel each other less strongly, such that the dual order parameter $|\Phi|^2$ shrinks, causing the critical temperature or critical current to diminish. The applied electric current is as the applied field for a superconductor: it competes with the established order. At first, all applied current is expelled, showing purely insulating behaviour. But in the type-II regime detailed in this chapter, above the *lower critical current* I_{c1}, vortex lines of current will be created. The current starts to penetrate in multiples of the current quantum I_0, until it is so large that the Mott order is completely destroyed. This point we call the *upper critical current* I_{c2}. It should not be confused with the critical current in a superconductor, which destroys superconducting order by inducing a too high magnetic field.

As opposed to the superconducting side, in the ‘atomic’ or infinite strong-coupling limit $U/J \to \infty$, there is no way in which the Mott insulating order can be perturbed. As such, at least formally, the insulating behaviour should persist and no current vortex lines can be formed. This could be characterized as the ‘type-I’ regime of the Mott insulator. Moreover, within the limits of validity of the model, this insulator will not be destroyed at any finite temperature. Therefore we have indicated a UV-cutoff in the phase diagram, above which our model is no longer descriptive. One could imagine for instance that the Cooper pairs will break up across this cut-off, so that there are no charged bosons to begin with.

This all seems quite straightforward, but it is actually profoundly surprising. In the regular XY-model, a 2-dimensional Bose-Mott insulator exists only at zero temperature, and it is destroyed at any finite temperature due to strong fluctuations (see e.g. [87, 88]). On the superfluid side there is still a finite-temperature Kosterlitz–Thouless transition because there the interactions are logarithmically long-range, but on the insulating side the dual gauge fields are massive. However the 3+1D Mott insulator at finite temperature is in the 4d XY universality class, and reverts basically to the mean-field result as it is at its upper critical dimension. The simple fact that there *is* a finite-temperature phase transition in a Bose-Mott insulator, even though it is just due to a higher dimensionality, is a novelty by itself.

5.5.3 Quantum critical regime

In this work we have not made any calculation at finite temperatures, and all our inferences for that regime stem from established knowledge. Actually, in the quantum disorder–temperature plane without applied field or current, this would just be the standard superconductor–Bose-Mott insulator quantum phase transition. Therefore, we expect a quantum critical point at zero temperature and associated quantum critical regime at finite temperature. The critical behaviour is also not part of this work.

Concurrently, it is not quite clear what happens at zero temperature when the applied field or current grows too large. For the superconductor one may still expect a transition to the normal state. However, the superconductor is destroyed by a large applied field because it induces a very large countercurrent. If the normal state is a Fermi liquid, and the Fermi liquid is intrinsically resistive, any current will immediately generate heat, making the assumption of zero temperature invalid. Similarly, if the 'normal' state is insulating as for instance in the underdoped cuprates, it is also hard to picture how a too large current can go over into insulating behaviour.

The situation is even more clear for the Bose-Mott insulator. Once the current permeating through the dual vortices gets too large, surely all of the insulator is destroyed. The current flowing is actually supercurrent: the vortex cores are locally superconducting as dictated by the duality. Therefore a large applied current should render the type-II Bose-Mott insulator into a superconductor. But the superconductor will be destroyed by a large current itself.

These considerations make us postpone a definite statement on the state of matter at zero temperature and large applied field or current. These regions are therefore indicated by a question mark *?* in the phase diagram.

5.5.4 Application to underdoped cuprates

We shall briefly map this general phase diagram onto the relevant phases of the underdoped cuprates (see figure 5.1). Surely, in real life things work differently than as pictured in the idealized scenario.

In the cuprates the quantum fluctuations are controlled by chemical doping, and it is therefore not possible to tune along the horizontal axis within one material sample. For each sample on the underdoped side, there is a thermal transition from the superconducting to the pseudogap state. But collecting data from several samples, there should also be an effective transition along the horizontal direction, which should therefore be governed by quantum fluctuations. The quantum critical point in the phase diagram of Fig. 5.3 does not appear as such in the cuprates—if at all present, many people believe a quantum critical point to be hidden by the superconducting 'dome', and it is actually related to the transition from the (doped) Mott insulating state to the Fermi liquid at large dopings, and probably of intrinsic fermionic nature.

Still, as we mentioned in §5.1.2, there is evidence for the pseudogap region to be a phase-disordered superconductor, and therefore a Bose-Mott insulator of repelling Cooper pairs. Thus, the transition (at a fixed finite temperature) from the superconductor to the pseudogap should be as the increasing quantum disorder transition of this chapter. Increasing quantum disorder is the increase of the fluctuations in the superconductor phase field. This suggests that the type-II Bose-Mott insulator may be found in the pseudogap region, and close to the phase transition to the superconductor, because there the Mott order parameter should be small, such that the dual penetration depth is large and vortices can be formed. This region is crudely indicated in Fig. 5.1.

5.6 Experimental signatures

In this chapter we have made a prediction for a new state of matter which we named "type-II Bose-Mott insulator". Whereas a regular (Mott) insulator would either completely expel electric current, or would finally permit current through dielectric breakdown like a capacitor, the type-II Mott insulator supports vortex lines of electric current such that it may penetrate at applied current much smaller than what would be required for complete breakdown. Furthermore, since the current lines form a (dual) Abrikosov lattice, the conductivity is very inhomogeneous.

Here we outline several experimental setups that may verify the existence of such type-II Mott insulating behaviour. Every time we assume that a clever experimentalist would be able to i) find a type-II Bose-Mott insulating material; ii) be able to make the samples as pictured; and iii) have the right experimental probes available and under full control. The experimental setups are sketched in figure 5.4.

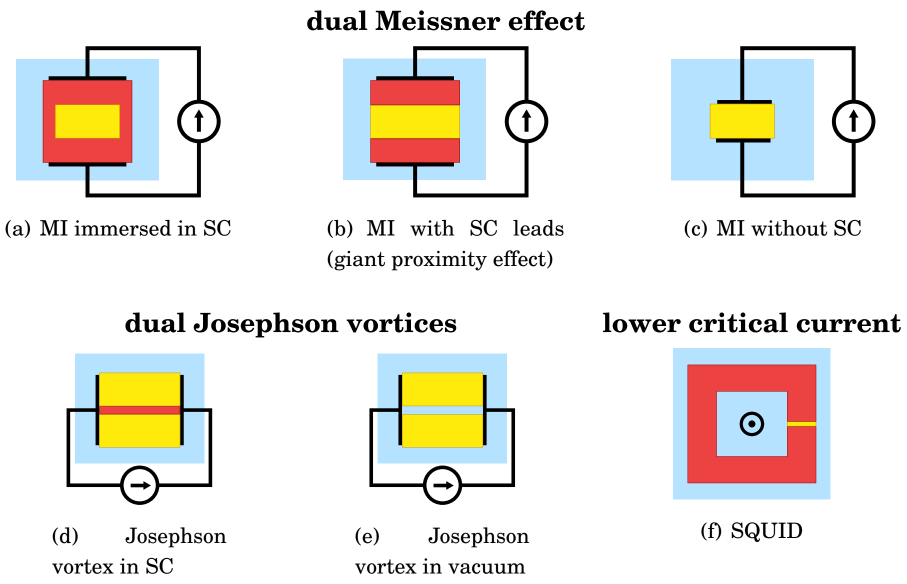
5.6.1 The vacua for electric current

Many effects in superconductivity appear at the boundary between the superconductor and empty space. These are both ground states or 'vacua' of their respective Hamiltonians. A magnetic field is free in empty space, but Meissner screened in the superconductor. These effects have to do with the Anderson–Higgs mechanism: photons are free in empty space but obtain a Higgs mass in the superconductor. In this regard, for the magnetic field also metals, dielectrics and so forth are like the vacuum, only with a different light velocity. The screening of photons in a metal is certainly not the Meissner effect, and the photons do not gain a mass even though they interact heavily with the electrons/quasiparticles. Most clearly, a static magnetic field can exist within a metal.

But for electric current, things are really different. We add a third vacuum: the type-II Bose-Mott insulator. As we have seen, electric current is to the Mott insulator as magnetic field is to the superconductor. Continuing the duality reasoning: the superconductor is to the Mott insulator as empty space is to the superconductor. What we mean is: an electric current is free in the superconductor (as long as it does not exceed the critical current) in the sense that a persistent current may run forever. But this current obtains a Higgs mass in the Bose-Mott insulator, just as the magnetic field does in a superconductor (Eq. (5.1.2)).

Conversely, the relation of empty space to the Bose-Mott insulator has no counterpart in the superconductor. As such, the situation is even richer, and more diverse tunnelling and/or junction experiments could be conceptualized. In figure 5.4, yellow is the type-II Mott-insulator, red is superconductor and blue is empty space.

Even more vacua are to be envisaged. Both the Bose-Mott insulator and



(a) MI immersed in SC
(b) MI with SC leads (giant proximity effect)
(c) MI without SC
(d) Josephson vortex in SC
(e) Josephson vortex in vacuum
(f) SQUID

Figure 5.4: Proposed experimental setups. The type-II Bose-Mott insulator (MI) is in yellow; the superconductor (SC) in red; and the empty space/Maxwell vacuum in blue. The circle and arrow represent a current source.
(a) The MI is completely immersed in SC which acts as the "current vacuum". The SC walls should be thin as to curtail the critical current. If current vortex lines will form, the total supercurrent will surpass this critical current. (b) A junction experiment with a thick MI layer between SC leads. This will only succeed if it is not necessary to have the current penetrate bit by bit from the outside, but may force it to form a vortex line from top to bottom immediately. This setup is used in the giant proximity effect (GPE). (c) Capacitor. Perhaps any current bias will cause vortex lines to form, even if it is not supercurrent. Then just MI between normal leads should short-circuit way before dielectric breakdown occurs. (d) Equivalent of Josephson vortices where the vortex line does not form inside MI but within a narrow junction layer of SC. (e) Perhaps the SC vacuum is unnecessary, and a dual Josephson vortex may even form in empty space. (f) SQUID setup in which current bias in increased in very small steps by a perpendicular magnetic field (circle with dot). Current will not flow until the first vortex is formed. This experiment measures the lower critical current.

the superconductor are made out of Cooper pairs, but in a normal metal those pairs are broken up. There simply are more building blocks, especially when making junction geometry setups.

5.6.2 Dual Meissner effect

The Mott insulator wants to expel current as all insulators do. However, when the applied current exceeds the lower critical current I_{c1}, dual vortices will form as lines of supercurrent, such that at least part of the current is permitted to flow through the material. Similar to showing the Meissner effect by measuring the magnetization of a block of superconducting material in the presence of a magnetic field, this dual Meissner effect for current may be demonstrated.

As explained in §5.6.1, the immediate analogue of the superconductor in an applied magnetic field is to immerse a block of type-II Mott insulator in superconducting material. This is because the common understanding is that the magnetic field lines penetrate from the outside to the centre to form the first vortex; similarly angular momentum in a superfluid travels from the outside in to form the first vortex. Now we have electric current which is not supported in the Maxwell vacuum but it is in the superconductor.

Therefore the first thing to try is pictured in Fig. 5.4(a). Current is induced to flow through the superconductor. Since current in a superconductor always flows near the boundary, it must be very narrow where the Mott insulator is immersed, presumably within one penetration depth λ. If the Mott insulator were perfectly insulating, the critical current J_c for this configuration would be limited by the narrow superconducting layers. But if vortices can form, some of the current will flow through the Mott insulator, leading to a much higher critical current.

It could also be that the current vortex lines will form even without superconducting leads, by just forcing regular, not super-, electric current through the Mott insulator. This is pictured in Fig. 5.4(c), and is in fact a capacitor. For a true insulator, current will only flow after dielectric breakdown, which only happens for really large currents. But if vortices form, that will happen at much lower current biases. There may be additional interface effects such as Andreev reflection, but it is our conviction that any form of electric current bias should suffice.

It may be that the vortices will only form if the applied current is a supercurrent. Then the leads should be made out of superconductor, as in Fig. 5.4(b). The signature will be the same: no current will flow for a good insulator, but current will soon flow for a type-II Mott insulator. In fact, this may have already been measured, related to the so-called giant proximity effect (GPE). Right after the high-temperature superconductors were discovered, people made junction setups to study their properties. An interesting type of junction is to have a superconductor of lower T_c sandwiched between two layers of material with higher T_c, and measure at a temperature right in between [89]. The question is whether the leads will induce superconductivity in the middle layer which is above its T_c. Surprisingly, a supercurrent was observed even in very thick layers, which does not conform to the regular Josephson effect, that is ultimately caused by the overlap of exponentially decaying wave functions. Even though there was some doubt related to the presence impurities, the final word seem to be that the effect is real [90]. It is also unexplained up to this day.

There is a proposal by Marchand *et al.* [91] that suggests that the phase rigidity of the superconducting leads prevents vortices to unbind in the middle layer, thus retaining the superconductivity even above T_c, not unlike the theme of this thesis. However, we suggest another mechanism: the formation of vortex lines of electric current. This automatically enables the middle layer to be very thick, because the energy cost of a vortex line grows as the length of the line, where as all other mechanisms affect the whole layer homogeneously, scaling as the volume. The telltale difference between our proposal and the earlier one, is that the vortex lines will show as an inhomogeneous distribution of conductivity, as opposed to homogeneous.

5.6.3 Dual Josephson vortices

In §4.4.3 we mentioned that, in superconductors, there can also be vortices in Josephson junctions, which are quantized but do no have a normal core and no core energy. This situation may be mimicked in the type-II Mott insulator. The influence of the Mott condensate will stray just beyond the edge of the material, so that in narrow gaps also vortices may arise. They are parallel to the edge of the Mott insulator.

Following the argument in §5.6.1 the immediate analogue of the Josephson vortex would be to have a very thin layer of superconductor between two pieces of type-II Mott insulator as in Fig. 5.4(d). Setting a current bias along the layer should cause the formation of vortices, manifested as a line of electric current. Of course, current flowing through a superconductor is nothing special, so this effect may not exist, or it may be very hard to detect. Still, the vortices are quantized, and therefore different in nature from regular supercurrent.

Another thing to try is to leave out the superconductor, and see whether a vortex can form under the influence of the Mott condensate wave function in the Maxwell vacuum (Fig. 5.4(e)). It is however hard to imagine how electric current would flow through empty space, and this is definitely not the first place to look for this effect. Other vacua such as a normal metal may also be interesting.

5.6.4 Lower critical current

Instead of trying to see the vortices directly, one could also attempt to determine when the first vortex is formed, that is: what is the value of the lower critical current I_{c1}? One advantage is that current is measurable to very high precision. We propose a superconducting quantum interference device (SQUID) setup as in figure 5.4(f). The (Josephson) junction in the SQUID is now made of type-II Mott insulator.

Applying a magnetic field perpendicular to the loop as indicated will induce a (persistent) current in the superconductor. Related to the phenomenon of flux quantization (§2.1.2), the magnetic field will only penetrate through the inner area when the current is actually allowed to flow. Reading out the amount of field that does get through, for instance by another SQUID, will tell how much current is flowing trough the loop. We envisage that, while increasing applied magnetic field, at first no current will flow until suddenly the first dual vortex will form and current does start to flow. The point of this jump is precisely the lower critical current I_{c1}. This should continue in a stepwise manner. Not only will this quantitatively determine the value of this parameter, but the sudden jump and ladder pattern are also qualitatively different from regular Josephson junctions.

5.6.5 Inhomogeneous conductivity

In many of the proposed setups in figure 5.4, the type-II behaviour of the Bose-Mott insulator would show in the inhomogeneity of the conductivity.

The typical length scale is the lattice spacing of the dual Abrikosov lattice, which depends the dual penetration depth and the amount of vortices related to the magnitude of the applied current and the size of the current quantum I_0. These parameters in turn depend on the "strength" of the Mott condensate Φ_∞^2, which varies from material to material and presumably also with temperature. This should be calculated or measured on an individual basis. The inhomogeneity itself is however a strong qualitative prediction.

Another problem is that the dual penetration depth $\tilde{\lambda}$ may typically be quite large (see §5.4.3). Presumably, following intuition from regular Abrikosov vortex physics, this would imply that the vortex lines reside quite deep below the edge of the Mott insulator, and all surface sensitive techniques would suffer from this complication.

Leaving these matters aside, there are several techniques that could measure the inhomogeneous conductivity. They should i) have high spatial resolution to see the current lines and the insulating regions in between; ii) have high conductivity resolution to measure the possibly low value of the current quantum I_0; and iii) be able to operate at temperatures low enough that the quantum phase transition dominates thermal fluctuations.

Scanning tunnelling spectroscopy (**STS**) is a very sensitive technique with extremely high spatial resolution. However it cannot probe further than several lattice spacings below the surface. Microwave Impedance Microscopy (**MIM**) directly measures the conductivity and up to 100nm resolution, but suffers the same surface limitations. Low energy electron microscopy (**LEEM**) measures the local electric field non-invasively, and for insulators should be able to do so up to a reasonable depth, and with high spatial resolution. A current problem is to cool the samples to a low enough temperature.

In the appendix 5.A we calculate the conductivity for both the superconducting and the Bose-Mott insulating phases.

5.6.6 Foreseeable complications

There are many possible complications in all of these proposals that may spoil a clean signature of the vortex current lines. It could be that the numbers simply do not work out. The current quantum I_0 seems rather large (§5.4.5), so that there will only be a few vortices deep below the surface. Or, the applied current necessary to induce the first vortex may exceed the superconductor critical current density J_c.

More importantly, most Mott insulators such as the underdoped cuprates are in fact poor insulators, meaning there will always be leak currents. This can be understood by considering figure 2.3: each excitation of the Mott insulator above the ground state immediately leads to free current carriers. As soon as the doublon and holon are formed, there is no further energy penalty for their hopping around. Therefore any experiment that relies on the distinction between insulating and conducting behaviour, and in particular the lower critical current setup of Fig. 5.4(f), has to deal with this drawback.

But the primary important effect to be expected is the strong pinning of the vortices. It is well known that the cuprates are in the 'dirty' limit where the coherence length is really short. We expect the same to hold for the Mott vortices. An Abrikosov vortex lattice can only exist because of pinning forces, since vortices in motion dissipate energy, and any fluctuation will cause such motion in an unpinned lattice. Indeed, the limiting factor in making high-field superconducting magnet coils is the ability to pin the vortices.

The pinning occurs on so-called pinning centres (impurities or defects), which are distributed unevenly throughout the material. Therefore the vortices follow the pinning centres rather than the vortex lattice, and the lines will most often not really be straight. These effects cause a large deviation from the idealized case. We expect similar behaviour for the Mott vortices. It may cause the vortex state to become 'glassy' and may in particular obscure the transition from the purely insulating to the vortex lattice state under applied current (at J_{c1}). Still, the strong non-linearity in the I–V characteristic should distinguish the type-II Mott insulator from a regular (doped) Mott insulator.

In all of our considerations, we have assumed the dual London limit $|\Phi|(x) = \Phi_\infty$ (no amplitude fluctuations). This should be good in the extreme type-II limit, but since this is all unexplored territory, one should keep a keen eye on a less robust condensate, which may have more obfuscated signatures.

5.7 Summary

We predict a new state of matter called "type-II Bose-Mott insulator". Just as in a type-II superconductor the Meissner effect expels magnetic field but permits it in the form of quantized vortex flux lines, this material normally expels electric current but permits it in the form of quantized vortex current lines. The current quantum is not fundamental, but depends on system-specific parameters. Otherwise, almost all the properties of type-II superconductors are mirrored, where magnetic field is to be replaced by electric current. All these features are collected in a new phase diagram (Fig. 5.3).

The current vortex lines may be found in cold atoms in optical lattices, arrays of Josephson junctions, but moreover in the pseudogap phase of underdoped cuprates, which are in this context fluctuating Bose-Mott insulators of incoherent Cooper pairs. We have proposed several experiments that may see the current lines (Fig. 5.4). If the type-II Mott behaviour is confirmed, this would be strong evidence of the pseudogap region as a phase-disordered superconductor.

There may be many ways in which this idealized picture can be complicated in nature. But since the study of Abrikosov vortices is over 50 years old and still going strong, we believe that with time the current vortex lines will show themselves just as clearly as their superconductor siblings.

5.A The conductivity of the superconductor and Bose-Mott insulator

Here we calculate the conductivity from the quantum partition sum as the response to an applied electric field. The conductivity σ in imaginary time τ is defined as,

$$\langle j_a(\mathbf{x},\tau)\rangle = \int \mathrm{d}^D\tilde{x}\mathrm{d}\tilde{\tau}\ \sigma_{ab}(\mathbf{x}-\tilde{\mathbf{x}},\tau-\tilde{\tau})E_b(\tilde{\mathbf{x}},\tilde{\tau}). \tag{5.33}$$

Here a and b are spatial vector indices; $\langle\ldots\rangle$ denotes expectation value. This equation defines the conductivity per spacetime volume, which has units of $\frac{\mathrm{C}^2}{\mathrm{Jsm}^{D-2}}\frac{1}{\mathrm{m}^D\mathrm{s}}$. The volume-integrated conductivity is related to the conductance as $g=\sigma A/l$ in D=3, where A is the area of the conductor, and l its length. This explains the factor $1/\mathrm{m}^{D-2}$ in the previous expression. It is the conductance which has the same units in any dimensions. The quantum of conductance, which features for example in the quantum Hall effect, is $\frac{\mathrm{e}^2}{\mathrm{h}}$.

The electric field can be expressed in terms of electromagnetic potentials,

$$\mathbf{E}(\mathbf{x},t) = -\nabla V(\mathbf{x},t) - \partial_t\mathbf{A}(\mathbf{x},t). \tag{5.34}$$

In our calculation, we will take the functional derivative of this expression with respect to A_a, with a spatial only. Therefore, we will disregard the term $\sim V$, since it will drop out anyway. When going to imaginary time $t \to \mathrm{i}\tau$, (5.34) will go over to,

$$\mathbf{E}(\mathbf{x},\tau) = \mathrm{i}\partial_\tau \mathbf{A}(\mathbf{x},\tau). \tag{5.35}$$

Substituting this expression in (5.33), performing a partial integration, and taking the functional derivative on both sides gives,

$$\begin{aligned}
\frac{\delta}{\delta A_c(y,\tau_y)}\langle j_a(\mathbf{x},\tau)\rangle &= \frac{\delta}{\delta A_c(y,\tau_y)}\int \mathrm{d}^D\tilde{x}\mathrm{d}\tilde{\tau}\ \big(-\mathrm{i}\partial_{\tilde{\tau}}\sigma_{ab}(\mathbf{x}-\tilde{\mathbf{x}},\tau-\tilde{\tau})\big)A_b(\tilde{\mathbf{x}},\tilde{\tau}) \\
&= \int \mathrm{d}^D\tilde{x}\mathrm{d}\tilde{\tau}\ \big(\mathrm{i}\partial_\tau\sigma_{ab}(\mathbf{x}-\tilde{\mathbf{x}},\tau-\tilde{\tau})\big)\delta_{bc}\delta(\tilde{x}-y) \\
&= \mathrm{i}\partial_\tau\sigma_{ac}(\mathbf{x}-\mathbf{y},\tau-\tau_y).
\end{aligned} \tag{5.36}$$

We can define the Fourier transform of the conductivity in terms of Matsubara frequencies ω_n and wave vectors $\mathbf{k}$ as follows,

$$\sigma_{ab}(\mathbf{k},\mathrm{i}\omega_n) = \int \mathrm{d}^D\tilde{x}\mathrm{d}\tilde{\tau}\ \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\tilde{\mathbf{x}}}\mathrm{e}^{-\mathrm{i}\omega_n\tilde{\tau}}\sigma_{ab}(\tilde{\mathbf{x}},\tilde{\tau}) = \int \mathrm{d}^d\tilde{x}\ \mathrm{e}^{-\mathrm{i}k\cdot\tilde{x}}\sigma_{ab}(\tilde{x}). \tag{5.37}$$

To get to the result of (5.36), multiply by the frequency,

$$\begin{aligned}
-\omega_n\sigma_{ab}(\mathbf{k},\mathrm{i}\omega_n) &= \int \mathrm{d}^d\tilde{x}\ \big(-\omega_n\mathrm{e}^{-\mathrm{i}k\cdot\tilde{x}}\big)\sigma_{ab}(\tilde{x}) = \int \mathrm{d}^d\tilde{x}\ \big(-\mathrm{i}\partial_{\tilde{\tau}}\mathrm{e}^{-\mathrm{i}k\cdot\tilde{x}}\big)\sigma_{ab}(\tilde{x}) \\
&= \int \mathrm{d}^d\tilde{x}\ \mathrm{e}^{-\mathrm{i}k\cdot\tilde{x}}\big(+\mathrm{i}\partial_{\tilde{\tau}}\sigma_{ab}(\tilde{x})\big).
\end{aligned} \tag{5.38}$$

Now substituting $\tilde{x} = x - y$ and noticing that $\partial_\tau f(\tau) = \partial_\tau f(\tau-\tau_y)$, we have indeed derived the Fourier transform of (5.36).

Now the current can be retrieved from the generating functional Z,

$$Z = \int \mathscr{D}\{\text{fields}\}\exp\big(-\frac{1}{\hbar}S_\mathrm{E}\big), \tag{5.39}$$

where S_E is the Euclidean action. It is,

$$\langle j_a(x)\rangle = -\hbar\frac{1}{Z[0]}\frac{\delta}{\delta A_a(x)}Z[A]. \tag{5.40}$$

Indeed, when one takes the action of a Ginzburg–Landau superconductor Eq. (2.34),

$$S_\mathrm{E} = \int \mathrm{d}^D x\mathrm{d}\tau\ -\frac{\hbar^2}{2m^*}\rho_\mathrm{s}\big(\partial_\mu^{\mathrm{ph}}\phi(x) - \frac{e^*}{\hbar}A_\mu^{\mathrm{ph}}(x)\big)^2, \tag{5.41}$$

one finds

$$\begin{aligned}\langle j_a(x)\rangle &= -\hbar(-\frac{1}{\hbar})(-\frac{\hbar^2}{m^*}\rho_{\rm s})(-\frac{e*}{\hbar})\big(\nabla_a\phi(x)-\frac{e^*}{\hbar}A_a(x)\big)\\ &= \frac{e^*\hbar}{m^*}|\Psi|^2\big(\nabla_a\phi(x)-\frac{e^*}{\hbar}A_a(x)\big),\end{aligned} \tag{5.42}$$

which agrees with Eq. (2.7).

Now from (5.40), (5.38) and (5.36) we find,

$$\omega_n\sigma_{ab}(\mathbf{k},\mathrm{i}\omega_n) = \int \mathrm{d}^d(x-y)\ \mathrm{e}^{-\mathrm{i}k(x-y)}\frac{\hbar}{Z[0]}\frac{\delta}{\delta A_b(y)}\frac{\delta}{\delta A_a(x)}Z[A]\big|_{A=0}. \tag{5.43}$$

The restriction $A = 0$ is taken, because we want to know the *linear* response of the (electron) system; when keeping A around, one also incorporates non-linear contributions.

5.A.1 Superconductor

We have derived the Euclidean action from the charged superfluid in (2.34),

$$S_{\rm E} = \int \mathrm{d}^Dx\mathrm{d}\tau\ -\frac{\hbar^2}{2m^*}\rho_{\rm s}(\partial_\mu^{\rm ph}\phi-\frac{e^*}{\hbar}A_\mu^{\rm ph})^2. \tag{5.44}$$

We also need to include the Maxwell term, which we will treat in the next section.

The temporal components involve a speed, but those components play no role in this calculation. Using this action in the generating functional, we find,

$$\begin{aligned}&\omega_n\sigma_{ab}(\mathbf{k},\mathrm{i}\omega_n)\\ &= \int \mathrm{d}(x-y)\mathrm{e}^{-\mathrm{i}k(x-y)}\frac{\hbar}{Z[0]}\frac{\delta}{\delta A_c(y)}(-\frac{1}{\hbar})(-\frac{\hbar^2}{m^*}\rho_{\rm s})(-\frac{e^*}{\hbar})\big(\nabla_a\phi(x)-\frac{e^*}{\hbar}A_a(x)\big)Z[A]\\ &= \int \mathrm{d}(x-y)\mathrm{e}^{-\mathrm{i}k(x-y)}\big[\frac{e^{*2}}{m^*}\rho_{\rm s}\delta_{ac}\delta(x-y)+\frac{e^{*2}}{m^{*2}}\rho_{\rm s}^2\hbar\langle\nabla_a\phi(x)\nabla_b\phi(y)\rangle\big].\end{aligned} \tag{5.45}$$

In the last term appears the velocity–velocity correlation function. This can be easily extracted from the generating functional in the Lorenz or the Coulomb gauge, where the photon fields decouple from the phase velocity $\nabla\phi$, and can be disregarded for this calculation. Adding an external source $\mathscr{J}_\mu$, the action to consider is,

$$S_{\rm E} = \int \mathrm{d}^Dx\mathrm{d}\tau\ -\frac{\hbar^2}{2m^*}\rho_{\rm s}\frac{1}{2}(\partial_\mu^{\rm ph}\phi)^2+\mathscr{J}_\mu\partial_\mu^{\rm ph}\phi. \tag{5.46}$$

Then,

$$\frac{\hbar^2}{Z[0]}\frac{\delta}{\delta\mathscr{J}_b(y)}\frac{\delta}{\delta\mathscr{J}_a(x)}Z[\mathscr{J}]\big|_{\mathscr{J}=0} = \frac{1}{Z[0]}\int\mathscr{D}\phi\ \nabla_a\phi(x)\nabla_b\phi(y)\mathrm{e}^{-1/\hbar S_\mathrm{E}}$$
$$= \langle\nabla_a\phi(x)\nabla_b\phi(y)\rangle. \tag{5.47}$$

Next, we complete the square in (5.46) and integrate out the phase field to find,

$$\begin{aligned}
S_\mathrm{E} &= \int \mathrm{d}^D x\mathrm{d}\tau\ \frac{\hbar^2}{2m^*}\rho_\mathrm{s}\phi(\partial^2_\mathrm{ph})\big(\phi - 2\frac{1}{\frac{\hbar^2}{m^*}\rho_\mathrm{s}\partial^2_\mathrm{ph}}\partial^\mathrm{ph}_\mu\mathscr{J}_\mu\big)\\
&= \int \mathrm{d}^D x\mathrm{d}\tau\ -\frac{1}{2}\frac{m^*}{\hbar^2\rho_\mathrm{s}}\partial^\mathrm{ph}_\mu\mathscr{J}_\mu\frac{1}{\partial^2_\mathrm{ph}}\partial^\mathrm{ph}_\nu\mathscr{J}_\nu\\
&= \int \mathrm{d}^D x\mathrm{d}\tau\,\mathrm{d}^d k\,\mathrm{d}^d k'\ \frac{1}{2}\frac{m^*}{\hbar^2\rho_\mathrm{s}}\mathscr{J}_\mu(k')\mathrm{e}^{\mathrm{i}k'x}\frac{\partial^\mathrm{ph}_\mu\partial^\mathrm{ph}_\nu}{\partial^2_\mathrm{ph}}\mathrm{e}^{\mathrm{i}kx}\mathscr{J}_\nu(k)\\
&= \int \mathrm{d}^4 k\ \frac{1}{2}\frac{m^*}{\hbar^2\rho_\mathrm{s}}\mathscr{J}_\mu(-k)\frac{k^\mathrm{ph}_\mu k^\mathrm{ph}_\nu}{k^2_\mathrm{ph}}\mathscr{J}_\nu(k).
\end{aligned}\tag{5.48}$$

Now, by definition,

$$\frac{\delta}{\delta\mathscr{J}_a(x)}\mathscr{J}_b(y) = \delta_{ab}\delta^{D+1}(y-x) = \delta_{ab}\int\mathrm{d}^{D+1}k\ \mathrm{e}^{\mathrm{i}k(y-x)}. \tag{5.49}$$

Expressing $\mathscr{J}_b(y)$ in Fourier decomposition, one finds,

$$\frac{\delta}{\delta\mathscr{J}_a(x)}\mathscr{J}_b(y) = \frac{\delta}{\delta\mathscr{J}_a(x)}\int\mathrm{d}^{D+1}k\ \mathrm{e}^{\mathrm{i}ky}\mathscr{J}_b(k) = \int\mathrm{d}^{D+1}k\ \mathrm{e}^{\mathrm{i}ky}\big[\frac{\delta}{\delta\mathscr{J}_a(x)}\mathscr{J}_b(k)\big]. \tag{5.50}$$

Comparing these two equations, one concludes,

$$\frac{\delta}{\delta\mathscr{J}_a(x)}\mathscr{J}_b(k) = \delta_{ab}\mathrm{e}^{-\mathrm{i}kx}. \tag{5.51}$$

Now we can evaluate the velocity–velocity correlation function:

$$\begin{aligned}
\langle\nabla_a\phi(x)\nabla_b\phi(y)\rangle &= \frac{\hbar^2}{Z[0]}\frac{\delta}{\delta\mathscr{J}_b(y)}\frac{\delta}{\delta\mathscr{J}_a(x)}\mathrm{e}^{\frac{-1}{\hbar}\frac{1}{2}\frac{m^*}{\hbar^2\rho_\mathrm{s}}\int\mathrm{d}^d k\ \mathscr{J}_\mu(-k)\frac{k^\mathrm{ph}_\mu k^\mathrm{ph}_\nu}{k^2_\mathrm{ph}}\mathscr{J}_\nu(k)}\\
&= \frac{1}{Z[0]}\frac{\delta}{\delta\mathscr{J}_b(y)}\frac{-m^*}{\hbar\rho_\mathrm{s}}\int\mathrm{d}^d k\ \mathrm{e}^{\mathrm{i}kx}\frac{k_a k^\mathrm{ph}_\nu}{k^2_\mathrm{ph}}\mathscr{J}_\nu(k)Z[\mathscr{J}]\\
&= \frac{-m^*}{\hbar\rho_\mathrm{s}}\int\mathrm{d}^d k\ \mathrm{e}^{\mathrm{i}k(x-y)}\frac{k_a k_b}{k^2_\mathrm{ph}}.
\end{aligned}\tag{5.52}$$

Inserting this into (5.45) one finds,

$$\omega_n \sigma_{ab}(\mathbf{k},\mathrm{i}\omega_n) = \int \mathrm{d}(x-y)\mathrm{e}^{-\mathrm{i}k(x-y)} \frac{e^{*2}}{m^*}\rho_\mathrm{s}\big[\delta_{ac}\delta(x-y) + \int \mathrm{d}^d k' \ \mathrm{e}^{\mathrm{i}k'(x-y)} \frac{k'_a k'_b}{k'^2_\mathrm{ph}}\big]$$
$$= \frac{e^{*2}}{m^*}\rho_\mathrm{s}\big[\delta_{ab} - \frac{k_a k_b}{\frac{1}{c^2_\mathrm{ph}}\omega_n^2 + \mathbf{k}^2}\big]. \tag{5.53}$$

Now we analytically continue to real time $\mathrm{i}\omega_n \to \omega + \mathrm{i}\eta$ and invoke the Sokhostsky formula,

$$\lim_{\eta\to 0} \frac{1}{\omega + \mathrm{i}\eta} = P(\frac{1}{\omega}) - \mathrm{i}\pi\delta(\omega). \tag{5.54}$$

we finally obtain,

$$\mathrm{Re}\big[\sigma_{ab}(\mathbf{k},\omega)\big] = \frac{e^{*2}}{m^*}\rho_\mathrm{s}\pi\delta(\omega)\big[\delta_{ab} - \frac{k_a k_b}{-\frac{1}{c^2_\mathrm{ph}}\omega^2 + \mathbf{k}^2 - \mathrm{i}\eta\,\mathrm{sgn}(\omega)}\big]. \tag{5.55}$$

We are especially interested in the zero-momentum conductivity. Taking the limit $\mathbf{k} \to 0$ with ω still finite the complex conductivity reads,

$$\sigma_{ab}(\mathbf{k}=0,\omega) = \frac{e^{*2}}{m^*}\rho_\mathrm{s}\delta_{ab}\big(\pi\delta(\omega) - \mathrm{i}\frac{1}{\omega}\big). \tag{5.56}$$

This agrees for the conductivity derived from the two-fluid Drude model, with the normal component vanishing, see e.g. [51, eq. 2.44]. The real part of the electric conductivity is peaked at zero frequency, this is the DC conductivity. The imaginary part has the standard form $\sim \frac{1}{\omega}$, valid at non-zero frequencies. It can also be found from invoking the Kramers–Krönig relation. Furthermore it is only valid for frequencies corresponding to energies below the gap; for higher energies pair-breaking events have to be taken into account as well, like in the Mattis–Bardeen model. Since we are deep within the superconductor $\rho_\mathrm{s} \gg 1$, the approximation is valid for a large range of frequencies.

5.A.2 Vacuum conductivity

If there is a need to include the Maxwell term, one can derive its conductivity contribution as follows,

$$\begin{aligned} S_{\mathrm{E,MW}} &= \int \mathrm{d}\tau \mathrm{d}^D x \; \frac{-1}{4\mu_0}(\partial^c_\mu A_\nu - \partial^c_\nu A_\mu)^2 \\ &= \int \mathrm{d}\tau \mathrm{d}^D x \; \frac{1}{2\mu_0} A_\mu \big((\partial^c)^2 \delta_{\mu\nu} - \partial^c_\mu \partial^c_\nu\big) A_\nu \\ &= \int \mathrm{d}\omega_n \mathrm{d}^D k \; \frac{-1}{2\mu_0} A_\mu(-k)\big((k^c)^2 \delta_{\mu\nu} - k^c_\mu k^c_\nu\big) A_\nu(k). \end{aligned} \tag{5.57}$$

Using this expression in the partition function, we find from (5.43),

$$\begin{aligned} \omega_n \sigma_{ab}(\mathbf{k}, \mathrm{i}\omega_n) &= \int \mathrm{d}(x-y) \mathrm{e}^{-\mathrm{i}k(x-y)} \frac{\hbar}{Z[0]} \frac{\delta}{\delta A_b(y)} \\ &\qquad \frac{1}{\mu_0 \hbar} \int \mathrm{d}\tilde{k} \mathrm{e}^{\mathrm{i}\tilde{k}x} \big((\tilde{k}^c)^2 \delta_{a\nu} - \tilde{k}^c_a \tilde{k}^c_\nu\big) A_\nu(k) Z[A] \\ &= \int \mathrm{d}(x-y) \mathrm{d}k \; \mathrm{e}^{-\mathrm{i}k(x-y)+\mathrm{i}\tilde{k}(x-y)} \frac{1}{\mu_0} \big((\tilde{k}^c)^2 \delta_{ab} - \tilde{k}^c_a \tilde{k}^c_b\big) \\ &= \frac{1}{\mu_0}\big((k^c)^2 \delta_{ab} - k^c_a k^c_b\big) = \frac{1}{\mu_0}\big((\frac{1}{c^2}\omega_n^2 + \mathbf{k}^2)\delta_{ab} - \mathbf{k}_a \mathbf{k}_b\big). \end{aligned} \tag{5.58}$$

By continuation to real time one finds,

$$\sigma_{ab}(\mathbf{k}, \omega) = \mathrm{i} \frac{1}{\mu_0} \frac{1}{\omega + \mathrm{i}\eta} \big((-\frac{1}{c^2}\omega^2 + \mathbf{k}^2)\delta_{ab} - \mathbf{k}_a \mathbf{k}_b\big). \tag{5.59}$$

In the limit $\mathbf{k} \to 0$ this reduces to, using (5.54),

$$\sigma_{ab}(\omega) = \varepsilon_0 \delta_{ab} \big(\mathrm{i}\omega + \pi \delta(\omega) \omega^2\big). \tag{5.60}$$

Clearly the second term vanishes for all ω, so the conductivity is purely imaginary, and $\sigma(\omega) = -\mathrm{i}\varepsilon_0 \omega$. This agrees with simple inspection of the Ampère–Maxwell law for $\mathbf{k} \to 0$,

$$0 = \frac{1}{\mu_0} \mathrm{i}\mathbf{k} \times \mathbf{B} \to \frac{1}{\mu_0} \nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \partial_t \mathbf{E} \to \mathbf{J} + \mathrm{i}\varepsilon_0 \omega \mathbf{E} \equiv \mathbf{J} - \sigma(\omega)\mathbf{E}. \tag{5.61}$$

The last step is the definition of the conductivity σ [cf. Eq. (5.33)].

5.A.3 Superconductor from dimensionless variables

For the sequel, it will be useful to repeat the calculation employing dimensionless variables as much as possible. First, we need to define the functional derivative. Take a dimensionless field $f(x)$ which is a function of the dimensionful coordinate x_μ. Then the functional derivative is

$$\frac{\delta}{\delta f(x)} f(y) = \delta^d(x-y). \tag{5.62}$$

The right hand side has dimension $1/[x]^d$, so that also $[\frac{\delta}{\delta f(x)}] = 1/[x]^d$. Therefore, one is led to equate

$$\frac{\delta}{\delta f(x)} = \frac{1}{a^d}\frac{\delta}{\delta f(x')} \tag{5.63}$$

where $x' = x/a$ is the dimensionless length and a the lattice constant.

From the Euclidean action (2.34),

$$S_\mathrm{E} = \int \mathrm{d}^D x \mathrm{d}\tau \; -Ja^{2-D}\frac{1}{2}(\partial^\mathrm{ph}_\mu \phi - \frac{e^*}{\hbar}A_\mu)^2. \tag{5.64}$$

the dimensionless action,

$$S'_\mathrm{E} = \int \mathrm{d}^D x' \mathrm{d}\tau' \; -\frac{Ja}{\hbar c_\mathrm{ph}}\frac{1}{2}(\partial'_\mu \phi - A'_\mu)^2, \tag{5.65}$$

is obtained by the substitutions,

$$x = ax' \qquad \tau = \frac{a}{c_\mathrm{ph}}\tau' \qquad A_\mu = \frac{\hbar}{ae^*}A'_\mu \qquad S_\mathrm{E} = \hbar S'_\mathrm{E} \tag{5.66}$$

Now for the conductivity (5.43),

$$\begin{aligned}\omega_n \sigma_{ab}(\mathbf{k}, \mathrm{i}\omega_n) &= \int \mathrm{d}(x-y)\mathrm{e}^{-\mathrm{i}k(x-y)}\frac{\hbar}{Z[0]}\frac{\delta}{\delta A_b(y)}\frac{\delta}{\delta A_a(x)}Z[A]\big|_{A=0} \\ &= \frac{a^{D+1}}{c_\mathrm{ph}}\int \mathrm{d}(x'-y')\mathrm{e}^{-\mathrm{i}k'(x'-y')}\frac{\hbar}{Z[0]} \\ &\qquad \left(\frac{c_\mathrm{ph}}{a^{D+1}}\right)^2\left(\frac{ae^*}{\hbar}\right)^2\frac{\delta}{\delta A'_b(y')}\frac{\delta}{\delta A'_a(x')}Z[A']\big|_{A'=0}\end{aligned} \tag{5.67}$$

This expression is generally valid after the substitutions (5.66), not just for the superconductor action (5.65). Still, for the superconductor one finds,

$$\frac{\delta}{\delta A'_b(y')}\frac{\delta}{\delta A'_a(x')}Z[A'] = \frac{Ja}{\hbar c_\mathrm{ph}}\delta_{ab}\delta(x'-y') + \left(\frac{Ja}{\hbar c_\mathrm{ph}}\right)^2\langle \partial'_a\phi(x')\partial'_b\phi(y')\rangle. \tag{5.68}$$

Following a procedure similar to (5.52), one finds the dimensionless version,

$$\langle \partial'_a\phi(x')\partial'_b\phi(y')\rangle = -\frac{\hbar c_\mathrm{ph}}{Ja}\int \mathrm{d}^{D+1}k'\mathrm{e}^{\mathrm{i}k'(x'-y')}\frac{k'_a k'_b}{k'^2}. \tag{5.69}$$

For the conductivity we then find,

$$\omega_n \sigma_{ab}(\mathbf{k}, \mathrm{i}\omega_n) = \frac{c_{\mathrm{ph}}}{a^{D+1}} \hbar \frac{Ja}{\hbar c_{\mathrm{ph}}} \frac{e^{*2} a^2}{\hbar^2} [\delta_{ab} - \frac{k'_a k'_b}{k'^2}] = J a^{2-D} \frac{e^{*2}}{\hbar^2} [\delta_{ab} - \frac{k_a k_b}{k^2}], \quad (5.70)$$

which agrees with (5.53), as in $D = 3$ we have $Ja^{2-D} = \hbar^2 \rho_{\mathrm{s}}/m^*$. One can now proceed to real time just as in the previous section.

5.A.4 Bose-Mott insulator

The Bose-Mott insulator is a condensate of phase-vortices. One must express the phase field ϕ in terms of dual gauge fields which couple to a dual Higgs field. Across the phase transition, the action is (5.15),

$$S'_{\mathrm{E}} = \int \mathrm{d}\tau' \mathrm{d}^3 x' \; \frac{1}{2} g(\epsilon_{\mu\nu\kappa\lambda} \partial'_\nu b'_{\kappa\lambda})^2 + \frac{1}{2} |\Phi|^2 (\frac{1}{2} \sum_\alpha \delta_{\alpha\kappa} \partial'^{\mathrm{ph}}_\lambda \phi - b'_{\kappa\lambda})^2$$
$$+ \epsilon_{\mu\nu\kappa\lambda} \partial'_\nu b'_{\kappa\lambda} A'_\mu - \frac{1}{4\mu} (\partial'^c_\mu A'_\nu - \partial'^c_\nu A'_\mu)^2. \quad (5.71)$$

Again, since the conductivity is a property of the medium, we can leave out the (vacuum) Maxwell term. We can then directly integrate out the dual gauge fields, yielding an expression quadratic in the photon field, which can be inserted in Eq. (5.67). Now we run into the standard problem for calculating propagators for gauge fields: the gauge invariant inverse propagator in the Lagrangian cannot be inverted, in essence because it is a transversal projector, and no projector but the unit matrix has an inverse. The solution is to fix the gauge, most conveniently using the Lorenz gauge. We had already assumed this gauge fix in going to Eq. (5.15).

The action simplifies considerably. The only catch is that in the end result, one should remember to impose the constraints $\partial'_\mu w'_\mu = 0$ and $\partial'_\mu A'_\mu = 0$ by inserting the transversal projector $\delta_{\mu\nu} - k'_\mu k'_\nu / k'^2$ in the numerator. We denote with a ˜ components that are Lorenz-gauge fixed. Then

$$(\epsilon_{\mu\nu\kappa\lambda} \partial'_\nu b'_{\kappa\lambda})^2 = -b'_{\mu\lambda} (\partial'^2 \delta_{\mu\nu} - 2\partial'_\mu \partial'_\nu) b'_{\nu\lambda} \to -\tilde{b}'_{\kappa\lambda} \partial'^2 \tilde{b}'_{\kappa\lambda}. \quad (5.72)$$

Also the condensate mode $\partial'^{\mathrm{ph}}_\mu \chi$ does not couple to the dual gauge field b' and does therefore not contribute to the photon correlation function. The Higgs term is then simply $\frac{1}{2}|\Phi|^2 (\tilde{b}'_{\kappa\lambda})^2$. We are now in a position to integrate out the dual gauge field,

$$
\begin{aligned}
&\int \mathrm{d}\tau'\mathrm{d}^3x' \; -\frac{1}{2}g\tilde{b}'_{\kappa\lambda}\partial'^2\tilde{b}'_{\kappa\lambda}+\frac{1}{2}|\Phi|^2(\tilde{b}'_{\kappa\lambda})^2+\epsilon_{\mu\nu\kappa\lambda}\partial'_\nu b'_{\kappa\lambda}A'_\mu \\
&=\int \mathrm{d}\tau'\mathrm{d}^3x' \; \frac{1}{2}\mathcal{G}^{-1}\tilde{b}'_{\kappa\lambda}\left[\tilde{b}'_{\kappa\lambda}+2\mathcal{G}\epsilon_{\kappa\lambda\nu\mu}\partial'_\nu A'_\mu\right] \\
&=\int \mathrm{d}\tau'\mathrm{d}^3x' \; \frac{1}{2}\mathcal{G}^{-1}\left[\tilde{b}'_{\kappa\lambda}+\mathcal{G}\epsilon_{\kappa\lambda\nu\mu}\partial'_\nu A'_\mu\right]^2-\frac{1}{2}\epsilon_{\kappa\lambda\nu\mu}\partial'_\nu A'_\mu\mathcal{G}\epsilon_{\kappa\lambda\sigma\rho}\partial'_\sigma A'_\rho \\
&\to\int \mathrm{d}\tau'\mathrm{d}^3x' \; \frac{1}{2}A'_\mu(\delta_{\mu\nu}\partial'^2-\partial'_\mu\partial'_\nu)\mathcal{G}A'_\nu.
\end{aligned}
\tag{5.73}
$$

Here we have defined the inverse dual gauge field propagator $\mathcal{G}^{-1}=-g\partial'^2+|\Phi|^2$. To calculate the correlation function, we should apply a Fourier transformation as in (5.48),

$$
\begin{aligned}
&\int \mathrm{d}\tau'\mathrm{d}^3x' \; \frac{1}{2}A'_\mu(\delta_{\mu\nu}\partial'^2-\partial'_\mu\partial'_\nu)(-g\partial'^2+|\Phi|^2)^{-1}A'_\nu = \\
&\int \mathrm{d}^{D+1}k' \; -\frac{1}{2}A'_\mu(-k')\frac{\delta_{\mu\nu}k'^2-k'_\mu k'_\nu}{gk'^2+|\Phi|^2}A'_\nu(k').
\end{aligned}
\tag{5.74}
$$

The conductivity is now obtained by inserting this in (5.67),

$$
\begin{aligned}
\omega_n\sigma_{ab}(\mathbf{k},\mathrm{i}\omega_n)&=\frac{\hbar c_{\mathrm{ph}}}{a^{D-1}}\frac{e^{*2}}{\hbar^2}\int \mathrm{d}^d(x'-y')\mathrm{e}^{-\mathrm{i}k'(x'-y')}\frac{1}{Z[0]}\frac{\delta}{\delta A'_b(y')}\frac{\delta}{\delta A'_a(x')}Z[A'] \\
&=\frac{\hbar c_{\mathrm{ph}}}{a^{D-1}}\frac{e^{*2}}{\hbar^2}\int \mathrm{d}^d(x'-y')\mathrm{d}\bar{k}' \; \mathrm{e}^{-\mathrm{i}(k'+\bar{k}')(x'-y')}\frac{\delta_{ab}\bar{k}'^2-\bar{k}'_a\bar{k}'_b}{g\bar{k}'^2+|\Phi|^2} \\
&=\frac{1}{g}\frac{\hbar c_{\mathrm{ph}}}{a^{D-1}}\frac{e^{*2}}{\hbar^2}\frac{\delta_{ab}k'^2-k'_a k'_b}{k'^2+|\Phi|^2/g}=\frac{e^{*2}\rho_{\mathrm{s}}}{m^*}\frac{\delta_{ab}k^2-k_a k_b}{k^2+|\Phi|^2/ga^2}
\end{aligned}
\tag{5.75}
$$

In the last step we reverted to dimensionful units. In the limit $|\Phi|^2\to 0$ this reduces to the result for the superconductor (5.70). We are interested in the DC and AC conductivity, and therefore take the limit $\mathbf{k}\to 0$, to find,

$$
\sigma_{ab}(\mathrm{i}\omega_n)=\frac{e^{*2}\rho_{\mathrm{s}}}{m^*}\frac{1}{\omega_n}\frac{\delta_{ab}\omega_n^2}{\omega_n^2+c_{\mathrm{ph}}^2|\Phi|^2/ga^2}\equiv\frac{e^{*2}\rho_{\mathrm{s}}}{m^*}\delta_{ab}\frac{\omega_n}{\omega_n^2+M^2}. \tag{5.76}
$$

Here we defined $M^2=\frac{c_{\mathrm{ph}}^2}{a^2}\frac{|\Phi|^2}{g}$. We continue to real time by $\mathrm{i}\omega_n\to\omega+\mathrm{i}\eta$, where $\eta>0$. Then,

$$
\begin{aligned}
\sigma_{ab}(\omega)&=\frac{e^{*2}\rho_{\mathrm{s}}}{m^*}\delta_{ab}\frac{-\mathrm{i}\omega}{-\omega^2-2\mathrm{i}\omega\eta+M^2} \\
&=\frac{e^{*2}\rho_{\mathrm{s}}}{m^*}\delta_{ab}\frac{-\mathrm{i}\omega}{\big((\omega-M)+\mathrm{i}\eta\big)\big((-\omega-M)+\mathrm{i}\eta\big)}.
\end{aligned}
\tag{5.77}
$$

Clearly there are poles at $\omega = M - \mathrm{i}\eta$ and $\omega = -M + \mathrm{i}\eta$. Using the Sokhostsky formula Eq. (5.54) for the pole near $\omega = M$ we find for the real part of the conductivity,

$$\mathrm{Re}\,\sigma_{ab}(\omega) = Ja^{2-D}\frac{e^{*2}}{\hbar^2}\delta_{ab}\frac{-\mathrm{i}\omega}{-\omega - M}\big(-\mathrm{i}\pi\delta(\omega - M)\big) = Ja^{2-D}\frac{e^{*2}}{\hbar^2}\delta_{ab}\frac{\pi}{2}\delta(\omega - M). \tag{5.78}$$

We can conclude that there are gapped poles at $\omega = \pm M = \pm\frac{c_{\mathrm{ph}}}{a}\sqrt{|\Phi|^2/g}$, and the pole strength of each is half of that of the superconductor [cf. Eq (5.56)].

Chapter 6

Emergent gauge symmetry and duality

Breaking symmetry is easy but making symmetry is hard: this wisdom applies to global symmetry but not to local symmetry. The study of systems controlled by emergent gauge symmetry has become a mainstream in modern condensed matter physics. Although one discerns as a fundamental gauge symmetry only electromagnetism in the ultraviolet of condensed matter physics, it is now very well understood that in a variety of circumstances gauge symmetries that do not exist on the microscopic scale control the highly collective physics on the macroscopic scale. An intriguing but unresolved issue is whether the gauge structures involved in the Standard Model of high energy physics and perhaps even general relativity could be of such an emergent kind.

Up to now we have not focussed much on the emergence of gauge symmetries—rather we have taken them for granted as a either an unrelated coincidence or as a logical but still auxiliary tool in the vortex duality. This chapter discusses some of the deeper, underlying gauge principles that not only facilitate understanding the nature of the disordering transition, but even provide a new viewpoint to gauge symmetry in general, possibly adding to our comprehension of its importance.

A gauge symmetry is said to be emergent when it is not present in the microscopic model of the constituent particles or fields, but arises in the effective theory as a collective degree of freedom. We have encountered two examples in this work:

1. the "stay-at-home" gauge invariance associated with (doped) Mott insulators, expressing a local conservation law (see §2.3.4);

2. the global-to-local symmetry correspondence in the strong/weak (i.e. Kramers–Wannier, S-) dualities, or the expression of the Goldstone mode as a dual gauge field (see §§2.4.2, 3.1).

In the common perception these appear as quite different. Here we clarify that at least in the context of bosonic physics they are actually closely related. In fact, these highlight complementary aspects of the vacuum structure, and it just depends on whether one views the vacuum either using the canonical/Hamiltonian language (stay-at-home) or field-theoretical/Lagrangian (local–global duality) language.

In this chapter we shall first go through the Bose-Hubbard model/vortex–boson duality again in §6.1, emphasizing the dual aspects of the emerging gauge symmetries. The ability to switch back and forth between Hamiltonian and Lagrangian viewpoints yields some entertaining vistas on this well-understood theory. In particular, the condensate of vortices is to be understood as a coherent superposition of all possible vortex configurations, and we will show that this is completely equivalent as adding gauge symmetry to phase correlations.

To make the case that it can yield new insight, we apply it in §6.2 to the less familiar context of dualities in quantum elasticity. This deals with the description of quantum liquid crystals in terms of dual condensates formed from the translational topological defects (dislocations) associated with the fully ordered crystal. Using the Lagrangian language it was argued that such quantum nematics are equivalent to (linearized) Einstein gravity [43]. Here we will demonstrate that this is indeed controlled by the local symmetry associated with linearized gravity: translations are gauged, turning into infinitesimal Einstein transformations.

6.1 Vortex duality versus Bose-Mott insulators

A mainstream of the gauge theories in condensed matter physics dates back to the late 1980s when the community was struggling with the fundamentals of the problem of high-T_c superconductivity. It was recognized early on that this has to do with doping the parent Mott insulators and this revived the interest in the physics of the Mott insulating state itself [58, 92–94]. The point of departure is the Hubbard model for electrons,

$$H_{\rm FH} = -t \sum_{<ij>\sigma} (\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma} + \hat{c}^\dagger_{j\sigma}\hat{c}_{i\sigma}) + U\sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}\,, \tag{6.1}$$

describing fermions $\hat{c}^\dagger_{i\sigma}$ on site i with spin σ, hopping on a lattice with rate t, subjected to a strong local Coulomb interaction U. Here $\hat{n}_{i\sigma} = \hat{c}^\dagger_{i\sigma}\hat{c}_{i\sigma}$ is the fermion number operator. A much simpler problem is the Bose-Hubbard model of §2.3. It describes spinless bosons created by $\hat{b}^\dagger_i$ hopping on a lattice with a rate t subjected to an on-site repulsion U,

$$H_{\rm BH} = -t \sum_{<ij>} \hat{b}^\dagger_i \hat{b}_j + U\sum_i \hat{n}^2_i\,. \tag{6.2}$$

Again $\hat{n}_i = \hat{b}^\dagger_i\hat{b}_i$ is the boson number operator. We assume in the remainder that the system is at "zero chemical potential", meaning that on average there is an integer number of fermions or bosons n_0 per site.

6.1.1 Stay-at-home gauge symmetry

Now these models are invariant under a *global* symmetry,

$$\hat{c}^\dagger_{i\sigma} \to \hat{c}^\dagger_{i\sigma} e^{i\alpha_\sigma} \qquad \text{or} \qquad \hat{b}^\dagger_i \to \hat{b}^\dagger_i e^{i\alpha}, \tag{6.3}$$

where the symmetry transformation is a scalar variable α that is constant for all lattice sites. But in the limit $U/t \to \infty$ the hopping term vanishes, and this symmetry is promoted to a local symmetry,

$$\begin{aligned} \hat{c}^\dagger_{i\sigma} &\to \hat{c}^\dagger_{i\sigma} e^{i\alpha_{i\sigma}}\,, & & & \hat{b}^\dagger_i &\to \hat{b}^\dagger_i e^{i\alpha_i} \\ \hat{c}_{i\sigma} &\to e^{-i\alpha_{i\sigma}}\hat{c}_{i\sigma}\,, & &\text{or} & \hat{b}_i &\to e^{-i\alpha_{i\sigma}}\hat{b}_i\,, \\ \hat{n}_i = \sum_\sigma \hat{c}^\dagger_{i\sigma}\hat{c}_{i\sigma} &\to \hat{n}_i & & & \hat{n}_i = \hat{b}^\dagger_i\hat{b}_i &\to \hat{n}_i. \end{aligned} \tag{6.4}$$

One discovers that a gauge symmetry emerges which controls the physics at long distances, while it is non-existent at the microscopic scale. This is the point of departure of a mainstream school of thought in condensed matter physics. In the fermionic model, there is still a dynamical spin system at work at low energies. Using various "slave-constructions" it was subsequently argued that quantum spin liquids characterized by fractionalized excitations can be realized when the resulting compact $U(1)$ gauge theory would end up in a deconfining regime. Conversely, the spinless Bose variety is completely featureless since it does not seem to break a manifest symmetry while low energy degrees of freedom are absent.

Note that at any finite U/t this gauge symmetry would be strictly broken; still at large values of this parameter it is a good idea to start from the gauge-invariant ground state, with deviation from this state entering as excitations.

The complete Hubbard models are defined in term of particle creation and annihilation operators, but in the Mott insulating state, the number of particles is *locally* conserved, i.e. conserved at each site separately, and only the number operator is present in the resulting Hamiltonian. The emergence of the gauge symmetry is caused by this local number conservation. One could picture that the annihilation–creation combination $c_i^\dagger c_i$ is now "tied" by emergent gauge bosons as force carriers: the particles are told to "stay at home". Indeed, the doped Mott insulator is described in such terms, leading to spin-charge separation and so forth.

This emergent gauge symmetry is not restricted to lattice models. Take for instance the effective Landau model describing the superfluid Eq. (2.1),

$$H = \int \mathrm{d}^3x \; \frac{1}{2}\tau|\nabla\Psi|^2 + \frac{1}{2}\alpha|\Psi|^2 + \frac{1}{4}\beta|\Psi|^4. \tag{6.5}$$

We have inserted a parameter τ for convenience. This model is invariant under global $U(1)$ symmetry $\Psi(x) \to \mathrm{e}^{\mathrm{i}\alpha}\Psi(x)$, where α is constant in space. But if we were to suppress the fluctuation term $\tau \to 0$, then this would be promoted to a local symmetry $\alpha \to \alpha(x)$. In other words, in the absence of fluctuations of the order parameter, the superfluid is indistinguishable from a superconductor. Furthermore the rigidity of the order parameter is now no longer enforced by a Goldstone mode, but by a local conservation law.

6.1.2 Vortex–boson duality

As detailed in §2.3, the Bose-Hubbard model at zero chemical potential can be mapped onto the XY- or phase-only model, which in turn maps onto the superfluid in the weak-coupling and continuum limit. We saw in §2.4 and chapter 3 that the phase transition to the Mott insulator is then formulated by the proliferation of topological defects, in this case vortices.

We needed to pass from the Goldstone field φ to its canonical conjugate, the supercurrent w_μ. This is the Noether current of the global symmetry $\varphi \to$

$\varphi + \varepsilon$. The smoothness of the Goldstone field ensures that the supercurrent is conserved $\partial_\mu w_\mu = 0$, which can be enforced by expressing it in terms of a dual gauge field,

$$w_\mu = \epsilon_{\mu\nu\lambda_1 \cdots \lambda_{d-2}} \partial_\nu b_{\lambda_1 \cdots \lambda_{d-2}}, \tag{6.6}$$

which is invariant under non-compact $U(1)$ gauge transformations,

$$b_{\lambda_1 \cdots \lambda_{d-2}} \to b_{\lambda_1 \cdots \lambda_{d-2}} + \partial_{[\lambda_1} \varepsilon_{\lambda_2 \cdots \lambda_{d-2}]}, \tag{6.7}$$

where $\varepsilon_{\lambda_2 \cdots \lambda_{d-2}}$ is any smooth $d-3$-form field. These gauge fields have the natural interpretation as the force carriers of the interactions between vortex excitations.

Once again, the global symmetry of the original model seems to be promoted to a local symmetry, but surely this non-compact symmetry of the Coulomb or superfluid phase is completely different from the compact $U(1)$ of the stay-at-home gauge invariance of the Higgs or Mott insulating phase.

The next step is to consider what happens across the phase transition. The vortices proliferate into a 'tangle of vortex world lines' or 'string foam', which is as a fluid medium minimally coupled to the dual gauge fields, which therefore undergo an Anderson–Higgs mechanism. The long-range correlations mediated by massless gauge fields now turn short-range.

6.1.3 The vortex condensate generates stay-at-home gauge

Up to this point we have just collected and reviewed some well-known results on phase dynamics. However, at first sight it might appear as if the matters discussed in the two previous subsections are completely unrelated. Departing from the Bose-Hubbard model the considerations of the previous subsection leave no doubt that in one or the other way the dual vortex '$d-2$-form superconductor' can be adiabatically continued all the way to the strongly coupled Bose-Mott insulator of the first subsection. The standard way to argue this is by referral to the excitation spectrum. The Bose-Mott insulator is characterized by a mass gap $\sim U$ (at strong coupling), and a doublet of "holon" (vacancy) and "doublon" (doubly-occupied site) propagating excitations being degenerate at zero chemical potential (see §§2.3,2.4.5 and 3.2). The vortex superconductor is a relativistic $U(1)/U(1)$ Higgs condensate characterized by a Higgs mass (a gap) above which one finds a doublet massive gauge bosons. In this regard there is a precise match. However, in the canonical formalism one also discovers the emergent $U(1)$ invariance associated with the sharp quantization of local number density in the Mott insulator. What has happened to this important symmetry principle in the vortex superconductor?

The answer is: the emergent compact $U(1)$ gauge symmetry of the Mott insulator is actually a generic part of the physics of the relativistic superconductor.

The argument is amazingly simple. The stay-at-home gauge does not show up explicitly in the Higgsed action describing the dual vortex condensate, for the elementary reason that all the quantities in this action are associated with the vortices which are in turn in a perfect non-local relation with the original phase variables. However, we know precisely what this dual superconductor is in terms of those phase variables. We can resort to a first quantized, world line description of the vortex superconductor, putting back "by hand" the phase variables. This constitutes a tangle of world lines of vortices, warping the original phases, and eventually we can even map that back to a first quantized wave function written as a coherent superposition of configurations of the phase field. To accomplish this in full one needs big computers [32, 33], but for the purposes of scale and symmetry analysis the outcomes are obvious.

The penetration depth λ_V of the dual vortex superconductor just coincides with the typical distance between vortices. At distances much shorter than λ_V the vortices do not scramble the relations between the phases at spatially separated points and at these scales the system behaves as the ordered superfluid,

$$\langle b^\dagger(r) b(0)\rangle \to \text{constant}\,, \quad r \ll \lambda_V\,, \tag{6.8}$$

However, at distances of order λ_V and larger, the vacuum turns into a coherent quantum superposition of "Schrödinger cat states" where there is either none, or one, or whatever number of vortices in between the two points 0 and r whose correlation of the phases of bosons we wish to know, see Fig. 6.1. We have arrived at exposing the simple principle which is the central result of this chapter: *since the vortex configurations are in coherent superposition, the phases acquire a full compact $U(1)$ gauge invariance.* Here is how to understand the physical concept: focus on the direction of the phase at the origin and look at the phase arrow at some distance point r. Consider a particular configuration of the vortices, and in this realization the distant phase

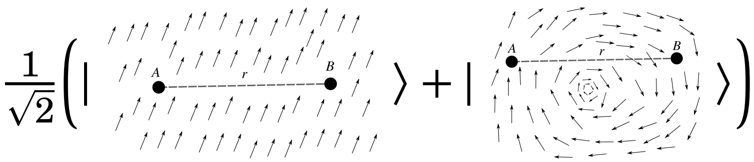


Figure 6.1: In the vortex condensate the correlation of the phase between a point A and another point B a distance r apart is in a superposition of having zero, one or any number of vortices in between. As such the phase at B with respect to that at A is completely undefined: it has acquired a full gauge invariance in the sense that any addition to the phase is an equally valid answer

will point in some definite direction which will be different from the phase at the origin as determined by the particular vortex configuration. However, since all different vortex configurations are in coherent superposition and therefore "equally true at the same time", all orientations of the phase at point r are also "equally true at the same time" and this is just the precise way to formulate that a compact $U(1)$ gauge symmetry associated with ϕ has emerged at distances λ_{V}.

The implication is that via Eq. (6.4) the emerging stay-at-home gauge invariance implies that in the Higgs condensate the number density associated with the bosons condensing in the dual superfluid becomes locally conserved on the scale λ_V. The Mottness therefore sets in only at scales larger than this λ_V. Notice that this mechanism does in fact not need a lattice: it is just a generic property of the field theory itself, which is independent of regularization. In fact, the seemingly all important role of the lattice in the standard reasoning in condensed matter when dealing with these issues is a bit of tunnel vision. It focusses on the strong-coupling limit where for large U, $\lambda_{\text{V}} \to a$, the lattice constant. However, upon decreasing the coupling strength, the stay-at-home gauge emerges at an increasingly longer length scale λ_{V}, to eventually diverge at the quantum phase transition. Close to the quantum critical point the theory has essentially forgotten about the presence of the lattice, just remembering that it wants to conserve number locally which is the general criterion to call something an insulator. In fact, Mottness can exist without a lattice altogether. A relativistic superconductor living in a perfect 2+1d continuum is physically reasonable. Since duality works in both directions, this can be in turn viewed as a quantum disordered superfluid, where the number density associated with the bosons comprising the superfluid becomes locally conserved.

By inspecting closely this simple vortex duality we have discovered a principle which might be formulated in full generality as: *the coherent superposition of the disorder operators associated with the condensation of the disorder fields has the automatic consequence that the order fields acquire a gauge invariance associated with the local quantization of the operators conjugate to the operators condensing in the order field theory.* We suspect that this principle might be of use also in the context of dualities involving more complex field theories.

6.2 Quantum nematic crystals and emergent linearized gravity

To substantiate this claim, let us now inspect a more involved duality which is encountered in quantum elasticity, where the principle reveals the precise reasons for why quantum liquid crystals have dealings with general relativity. Einstein himself already suggested the metaphor that the spacetime of general relativity is like an elastic medium. Is there a more literal truth behind it? In recent years Hagen Kleinert has been forwarding the view that quite deep analogies exist between plastic media (solids with topological defects) and Einsteinian spacetime [41, 42]. There appears room for the possibility that at the Planck scale an exotic "solid" (the "world crystal") is present, turning after coarse graining into the spacetime that we experience.

It turns out that this subject matter has some bearing on a much more practical question: what is the general nature of the quantum hydrodynamics and rigidity of quantum liquid crystals? Quantum liquid crystals [82] are just the zero temperature versions of the classical liquid crystals found in computer displays. These are substances characterized by a partial breaking of spatial symmetries, while the zero temperature versions are at the same time quantum liquids. Very recently indications have been found for variety of such quantum liquid crystals in experiment [95–100]. In the present context we are especially interested in the "quantum smectics" and "quantum nematics" found in high-T_c cuprates [84, 96–98] which appear to be also superconductors at zero temperature. Such matter should be, at least in the long-wavelength limit, governed by a bosonic field theory, and this "theory of quantum elasticity" [40, 45, 83, 101] is characterized by dualities that are richer, but eventually closely related to the duality discussed in the previous section.

Departing from the quantum crystal, the topological agents which are responsible for the restoration of symmetry are the dislocations and disclinations. The disclinations restore the rotational symmetry and the topological criterion for liquid crystalline order is that these continue to be massive excitations. The dislocations restore translational symmetry, and these are in crucial regards similar to the vortices of the previous section. In direct analogy with the Mott insulator being a vortex superconductor, the superconducting smectics and nematics can be universally viewed as dual "stress superconductors" associated with Bose condensates of quantum dislocations.

Using the geometrical correspondences of Kleinert [41, 42], arguments were put forward suggesting that the Lorentz-invariant version of the superfluid nematic in 2+1d is characterized by a low energy dynamics that is the same as at least linearized gravity [43]. Very recently it was pointed out that this appears also to be the case in the 3+1d case [102, 103]. A caveat is that Lorentz invariance is badly broken in the liquid crystals as realized in condensed matter physics. This changes the rules drastically and although the consequences are well understood in 2+1d [40, 45, 83] it remains to be clarified what this means for the 3+1d condensed matter quantum liquid crystals. The currently unresolved issue is how the gravitons of the 3+1d relativistic case imprint on the collective modes of the non-relativistic systems.

Here we want to focus on perhaps the most fundamental question one can ask in this context: although general relativity is not a Yang–Mills theory, it is uniquely associated with the gauge symmetry of general covariance or diffeomorphisms. Quite generally, attempts to identify "analogue" or "emergent" gravity in condensed matter systems have been haunted by the problem that general covariance is quite unnatural in this context. The gravity analogues currently contemplated in condensed-matterlike systems usually get as far as to identify a non-trivial geometrical parallel transport of the matter, that occurs in a "fixed frame" or "preferred metric" [104–111]; in other cases this issue of the mechanism of emerging general covariance is simply not addressed [112–114]. As we shall discuss, crystals are manifestly non-diffeomorphic. However, the relativistic quantum nematics appear to be dynamically similar to Einsteinian spacetime. For this to be true, in one way or another general covariance has to emerge in such systems. How does this work?

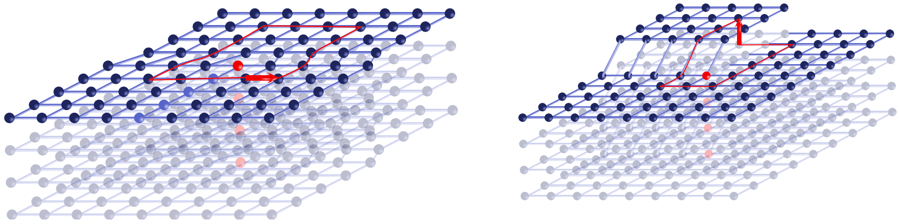
In close parallel with the vortex duality "toy model" of the previous section, we will explicitly demonstrate in this section that indeed general covariance is dynamically generated as an emergent IR symmetry. However, there is a glass ceiling: the geometry is only partially gauged. Only the infinitesimal "Einstein" translations fall prey to an emergent gauge invariance while the Lorentz transformations (rotations) remain in a fixed frame. This prohibits the inclusion of black holes and so forth, but this symmetry structure turns out to be coincident with the 'gauge fix' that is underlying *linearized* gravity. The conclusion is that relativistic quantum nematics constitute a medium that supports gravitons, but nothing else than gravitons.

For this demonstration we have to rely on the detour for the identification of the local symmetry generation as introduced in the previous section. Different from the Bose-Mott insulator, there is no formulation available for the quantum nematic in terms of a simple Hamiltonian where one can directly read off the equivalent of the stay-at-home gauge symmetry. We have therefore to find the origin of the gauging of the Einstein translations in the physics of the dislocation Bose condensate, but this will turn out to be a remarkably simple and elegant affair.

The remainder of this section is organized as the previous one. In section 6.2.1 we will first collect the various bits and pieces: a sketch of the way that "dislocation duality" associates the relativistic quantum nematic state with a crystal that is destroyed by a Bose condensate of dislocations. In section 6.2.2 we will subsequently review Kleinert's "dictionary" relating quantum elasticity and Einsteinian geometry, while at the end of this subsection we present the mechanism of gauging Einstein translations by the dislocation condensate. For simplicity we will focus on the 2+1d case; the generalities we address here apply equally well to the richer 3+1d case.

6.2.1 The quantum nematic as a dislocation condensate

Let us first introduce the field-theoretical side [40, 45, 83, 101]. The theory of quantum elasticity is just the 19th century theory of elasticity but now embedded in the Euclidean spacetime of thermal quantum field theory. To keep matters as simple as possible we limit ourselves to the Lorentz-invariant "world crystal", just amounting to the statement that we are dealing with a



(a) Edge dislocation (b) Screw dislocation

Figure 6.2: Dislocation lines (red spheres) in the relativistic 3D "world crystal" (two space and one time direction), formed by insertion of a half-plane of particles. Shown in red is the contour that measures the mismatch quantized in the Burgers vector (red arrow). If the Burgers vector is orthogonal to the dislocation line it is an edge location; if the Burgers vector is parallel it is a screw dislocation. In non-relativistic 2+1d there are only edge dislocations, since the Burgers vector is always purely spatial.

2+1d elastic medium being isotropic, both in space and time directions,

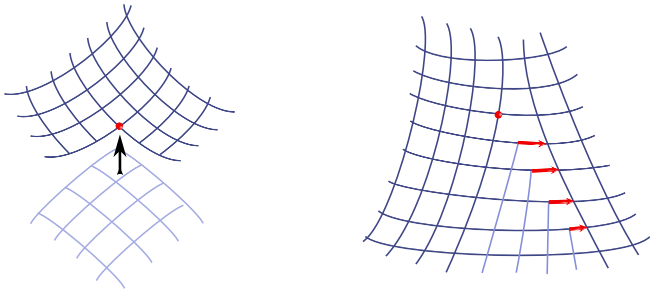
$$Z = \int Dw \ \mathrm{e}^{-S_{\mathrm{el}}} ,$$
$$S_{\mathrm{el}} = \int \mathrm{d}\tau \mathrm{d}x^2 \ \left[\mu w_{\mu\nu} w^{\mu\nu} + \frac{\lambda}{2} w_{\mu\mu}^2 \right] , \tag{6.9}$$

where,

$$w_{\mu\nu} = \frac{1}{2} \left(\partial_\mu u_\nu + \partial_\nu u_\mu \right) , \tag{6.10}$$

are the strain fields associated with the displacements u_ν of the "world crystal atoms" relative to their equilibrium positions. Here μ and λ are the shear modulus and the Lamé constant of the world crystal, respectively. At first view this looks like a straightforward tensorial generalization of the scalar field theory of the previous section. For the construction of the nematics one can indeed think about the displacements as "scalar fields with flavours" since this only involves the "Abelian sector" of the theory associated with translations. One should keep in mind however that one is breaking Euclidean space down to a lattice subgroup and this is associated with non-Abelian, infinite and semi-direct symmetry structure: the full theory beyond the dislocation duality is a much more complicated affair.

These issues become manifest when considering the topological defects: the dislocations and disclinations. The dislocation is the topological defect



(a) Disclination (b) Stack of dislocations

Figure 6.3: (a) 90° disclination in a square lattice. A wedge is inserted into a cut in the lattice. There is now one lattice point with five instead of four neighbouring sites (red); going along a contour around this point will result in an additional 90° rotation. The associated topological charge is the Frank vector, orthogonal to the plane and of size 90°. As the dislocation, in 2+1d spacetime the disclination point will trace out a world line. (b) Disclination as a stack of dislocations. Hence a disclination corresponds to a uniform polarization of Burgers vectors. As long as disclinations are massive, e.g. in the quantum nematic, dislocations appear only with balanced opposite Burgers vectors.

associated with the restoration of the translations. The dislocation can be viewed as the insertion of a half-plane of extra atoms terminating at the dislocation core. One immediately infers that it carries a vectorial topological charge: the Burgers vector indexed according to the Miller indices of the crystal. In 2+1 dimensions the dislocation is a particle (like the vortex) and as an extra complication the Burgers vector can either lie perpendicular [“edge dislocation”, Fig. 6.2(a)] or parallel [“screw dislocation”, Fig. 6.2(b)] to the propagation direction of its world line. The disclination is on the other hand associated with the restoration of the rotational symmetry. This can be obtained by the Volterra construction: cut the solid, insert a wedge and glue together the sides [see Fig. 6.3(a)]. This carries a vectorial charge (the Frank vector) as well. Finally dislocations and disclinations are not independent. On the one hand, the disclination can be viewed as a stack of dislocations with parallel Burgers vectors [Fig. 6.3(b)], while the dislocation can be viewed as a disclination–antidisclination pair displaced by a lattice constant.

Dislocations and disclinations do however have a distinguishable identity and this enables a tight, topological definition of quantum smectic and nematic order. A state where dislocations have spontaneously proliferated and condensed, while the disclinations are still massive, is a quantum liquid crystal. Since a disclination is coincident with a "uniform magnetization" of Burgers vectors, one cannot have a net density of parallel Burgers vectors as long as disclinations are suppressed [see Fig. 6.3(b)]. The Burgers vectors of the dislocations in the condensate have to be anti-parallel and therefore the dislocation breaks orientations rather than rotations, with the ramification that the order parameter is a director instead of a vector.

Finally, when all orientations of the Burgers vectors are populated equally in the condensate, one deals with a nematic breaking only space rotations. When only a particular Burgers vector orientation is populated one is dealing with a smectic because the translations are only restored in the direction of the Burgers vector: the system is in one direction a superfluid and in the other still a solid. To complete this outline, when the coupling constant is further increased there is yet another quantum phase transition associated with the proliferation of disclinations turning the system into an isotropic superfluid.

Let us now review the "dislocation duality": in close analogy with vortex duality, this shows how crystals and liquid crystals are related via a weak–strong duality. The requirement that disclinations have to be kept out of the vacuum is actually a greatly simplifying factor. One follows the same dualization procedure for the dislocations as for the vortices. Hence, we introduce Hubbard–Stratonovich auxiliary tensor fields $\sigma_{\mu\nu}$, rewriting the action as,

$$S = \int \mathrm{d}\tau \mathrm{d}x^2 \left[\frac{1}{4\mu} \left(\sigma_{\mu\nu}^2 - \frac{\nu}{1+\nu} \sigma_{\mu\mu}^2 \right) + \mathrm{i}\sigma_{\mu\nu} w_{\mu\nu} \right] , \tag{6.11}$$

where $\nu = \lambda/2(\lambda+\mu)$ is the Poisson ratio. We divide the displacement fields (having the same status as the phase field in vortex duality) in smooth and multivalued parts $u_\mu = u_\mu^{\mathrm{smooth}} + u_\mu^{\mathrm{MV}}$, and integrating out the smooth strains yields a constraint, in this case a Bianchi identity,

$$\partial_\mu \sigma_{\mu\nu} = 0 , \tag{6.12}$$

The physical meaning of $\sigma_{\mu\nu}$ is that they are the stress fields, which are conserved in the absence of external stresses as in Eq. (6.12): the above is just the stress–strain duality of elasticity theory. One now wants to parametrize the stress fields in terms of a gauge field. Since the stress tensor is symmetric this is most naturally accomplished in terms of Kleinert's double curl gauge fields,

$$\sigma_{\mu\nu} = \epsilon_{\mu\kappa\lambda}\epsilon_{\nu\kappa'\lambda'}\partial_\kappa\partial_{\kappa'}B_{\lambda\lambda'} \tag{6.13}$$

while the B's are *symmetric* tensors, otherwise transforming as $U(1)$ gauge fields.

To maintain the analogy with the vortex duality as tightly as possible, one can as well parametrize it in a normal gauge field, $\sigma_{\mu\nu} = \varepsilon_{\mu\kappa\lambda}\partial_\kappa b^\nu_\lambda$ with the requirement that one has to impose the symmetry of the stress tensor explicitly by Lagrange multipliers. Using this route one finds that the multivalued strains turn into a source term $ib^\nu_\mu J^\nu_\mu$ where,

$$J^{\mathrm{V}}_{\mu\nu} = \epsilon_{\mu\kappa\lambda}\partial_\kappa\partial_\lambda u^{\mathrm{MV}}_\nu \,, \tag{6.14}$$

This is just like a vortex current carrying an extra "flavour" ν. It is the dislocation current, where the flavour indicates the $D+1$ components of the Burgers vector. Like the vortices, dislocations have long-range interactions which are parametrized by the gauge fields b (or B), with the special effect that these are only active in the directions of the Burgers vectors.

The double curl gauge fields have the advantage that the symmetry is automatically built in while the "extra derivatives" enable the identification of the disclination currents. One finds,

$$S = \int \mathrm{d}\tau \mathrm{d}x^2 \, \left[\frac{1}{4\mu}\left(\sigma^2_{\mu\nu} - \frac{\nu}{1+\nu}\sigma^2_{\mu\mu}\right) + \mathrm{i}B_{\mu\nu}\eta_{\mu\nu} \right] \,, \tag{6.15}$$

where the "stress gauge fields" B are sourced by a total "defect current",

$$\begin{aligned} \eta_{\mu\nu} &= \epsilon_{\mu\kappa\lambda}\epsilon_{\nu\kappa'\lambda'}\partial_\kappa\partial_{\kappa'}w^{\mathrm{MV}}_{\lambda\lambda'} \,, \\ &= \theta_{\mu\nu} - \epsilon_{\mu\kappa\lambda}\partial_\kappa J_{\nu\lambda} \,, \end{aligned} \tag{6.16}$$

where $\theta_{\mu\nu}$ is the disclination current, and ν refers to the Franck vector component. The fact that the disclination current has "one derivative less" than the dislocation current actually implies that disclinations are in the solid confined—in the solid, a disclination is like a quark.

One now associates a much larger core energy to the disclinations than to the dislocations, and upon increasing the coupling constant a loop blowout transition will occur involving only the dislocation world lines—it is obvious from the single curl gauge field formulation that dislocations are just like vortices carrying an extra “Burgers flavour”. To obtain the quantum nematic one populates all Burgers directions equally and after some straightforward algebra one obtains the effective action for the “Higgsed stress photons” having the same status as Eq. (3.23) for the Mott insulator,

$$S = \int \mathrm{d}\tau \mathrm{d}x^2 \left[m_{\mathrm{nem}}^2 \sigma_{\mu\nu} \frac{1}{\partial^2} \sigma_{\mu\nu} + \frac{1}{4\mu} \left(\sigma_{\mu\nu}^2 - \frac{\nu}{1+\nu} \sigma_{\mu\mu}^2 \right) + \mathrm{i} B_{\mu\nu} \theta_{\mu\nu} \right] , \tag{6.17}$$

where σ should be expressed in the double curl gauge field $B_{\mu\nu}$ according to Eq. (6.13). In terms of the regular gauge fields b_μ^ν, the first term represents a Higgs mass, while the second term is like a Maxwell term. Nevertheless, in the nematic the disclinations still act as sources coupling to the double curl gauge fields.

Ignoring the disclinations, one finds in 2+1d that Eq. (6.17) describes a state is quite similar to a Mott insulator: all excitations are massive, and one finds now a triplet of massive “photons”. These are counted as follows: there are two propagating (longitudinal and transversal acoustic) phonons of the background world crystal, turning into “stress photons” after dualization and acquiring a mass in the nematic. In addition, the dislocation condensate adds one longitudinal stress photon.

As it turns out, the rules change drastically upon breaking the Lorentz invariance. In a crystal formed from material bosons, displacements in the time direction u_τ are absent, and this has among others the consequence that the dislocation condensate does not couple to compressional stress. Instead of the incompressible nature of the relativistic state, one finds now two massless modes in the quantum nematic: a rotational Goldstone boson associated with the restoration of the broken rotational symmetry, and a massless sound mode which can be shown to be just the zero sound mode of the superfluid. The non-relativistic quantum liquid crystals are automatically superfluids as well and their relation to gravity is obscured.

Turning to the 3+1d case one finds as extra complication that dislocations turn into strings and one has to address the fact that the “stress superconductor” is now associated with a condensate of strings. One meets the same complication as in vortex duality, which was tackled in chapter 3. The outcome is actually quite straightforward: the effective London actions of the type Eqs. (3.23),(6.17) have the same form regardless whether one deals with particle or string condensates, and these enter through the Higgs term $\sim \sigma^2/\partial^2$.

How to interpret the 2+1d relativistic quantum nematic? There are no low energy excitations and it only reacts to disclinations. It has actually precisely the same status as a flat Einsteinian spacetime in 2+1d that only feels the infinitesimal vibrations associated with gravitational events far away. Similarly, using the general relativity (GR) technology of the next section, it is also straightforward to demonstrate [102, 103] that in 3+1d one ends up with two massless spin-2 modes: the gravitons. To prove that it is precisely linearized gravity, let us consider next the rules of Kleinert that allow to explicitly relate these matters to gravitational physics.

6.2.2 Quantum elasticity field theory: the Kleinert rules

Elaborating on a old tradition in "mathematical metallurgy", Kleinert identified an intriguing portfolio of general correspondences between the field theory describing elastic media and the geometrical notions underlying general relativity. In order to appreciate what comes, we need to familiarize the reader with some of the entries of this dictionary. For an exhaustive exposition, see Kleinert's books on the subject [41, 42].

GR is a geometrical theory which departs from a metric $g_{\mu\nu}$, such that an infinitesimal distance is measured through,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x_\mu \mathrm{d}x_\nu \,, \tag{6.18}$$

One now insists that the physics is invariant under local coordinate transformations (general covariance) $x_\mu \to \xi_\mu(x_\nu)$; infinitesimal transformations then are like gauge transformations of the metric,

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \equiv g_{\mu\nu} + h_{\mu\nu} \,, \tag{6.19}$$

Only quantities are allowed in the theory which are invariant under these transformations and insisting on the minimal number of gradients, one is led to the Einstein–Hilbert action governing spacetime,

$$S = -\frac{1}{2\kappa}\int \mathrm{d}^D x \mathrm{d}t \, R\sqrt{-g} \,, \tag{6.20}$$

where $g = \det g_{\mu\nu}$ and R the Ricci scalar, while κ is set by Newton's constant. Together with the part describing the matter fields, the Einstein equations follow from the saddle points of this action.

How to relate this to solids? Imagine that one lives inside a solid and all one can do to measure distances is to keep track how one jumps from unit cell to unit cell. In this way one can define a metric "internal" to the solid, and the interesting question becomes: what is the fate of the diffeomorphisms ("diffs") Eq. (6.19)? In order to change the metric one has to displace the atoms and this means that one has to *strain* the crystal,

$$g_{\mu\nu} \to g_{\mu\nu} + w_{\mu\nu} , \tag{6.21}$$

But the strain fields are surely not gauge fields: the elastic energy Eq. (6.9) explicitly depends on the strain. Obviously, the crystal is non-diffeomorphic and it is characterized by a "preferred" or "fixed" frame. This is the deep reason that normal crystals have nothing to do with GR.

In standard GR the objects that are invariant keep track of curvature and these appear in the form of curvature tensors in the Einstein equations. Linearizing these, assuming only infinitesimal diffs as in Eq. (6.19), one finds for the Einstein tensor appearing in the Einstein equations, say in the 2+1d case to avoid superfluous labels,

$$G_{\mu\nu} = \epsilon_{\mu\kappa\lambda}\epsilon_{\nu\kappa'\lambda'}\partial_\kappa\partial_{\kappa'}h_{\lambda\lambda'} \tag{6.22}$$

One compares this with the disclination current Eq. (6.16) and one discovers that these are the same expressions after associating the strains $w_{\mu\nu}$ with the infinitesimal diffs $h_{\mu\nu}$. This is actually no wonder: at stake is that the property of curvature is independent of the gauge choice for the metric. One can visualize the curved manifold in a particular gauge fix, and this is equivalent to the fixed frame. The issue is that curvature continues to exist when one lets loose the metric in the gauge volume.

What is the meaning of the dislocation tensor? Cartan pointed out to Einstein that his theory was geometrically incomplete: one has to allow also for the property of torsion. It turns out that torsion is "Cartan-Einstein" GR sourced by spin currents and the effects of it turn out to be too weak to be observed (see e.g. Ref. [115]). In the present context, the torsion tensor appearing in the equations of motions precisely corresponds with the dislocation currents. With regard to these topological aspects, crystals and spacetime are remarkably similar.

However, given the lack of general covariance the dynamical properties of spacetime and crystals are entirely different. For obvious reasons, spacetime does not know about phonons while crystals do not know about gravitons, let alone about black holes. A way to understand why things go so wrong is to realize that the disclinations encode for curvature, and gravitons can be viewed as infinitesimal curvature fluctuations. As we already explained, disclinations are confined in crystals meaning that it costs infinite energy to create curvature fluctuations in normal solids.

Let us now turn to the relativistic quantum nematics: here the situation looks much better. Gravity in 2+1d is incompressible in the sense that the constraints do not permit massless propagating modes, the gravitons. We also found out that disclinations are now deconfined and they appear as sources in the effective action Eq. (6.17): this substance knows about curvature. In fact, one can apply similar considerations to the 3+1d case, where two massless spin-2 modes are present. The relativistic quantum nematic in 3+1d behaves quite like spacetime!

To make the identification even more precise, one notices that the expression for the linearized Einstein tensor Eq. (6.22) is coincident with the expression for the stress tensor in terms of the double curl gauge field $B_{\mu\nu}$, Eq. (6.13). But now one is dealing with gauge invariance both of $B_{\mu\nu}$ and $h_{\mu\nu}$ while they are both symmetric tensors. At least on the linearized level the stress tensor *is* the Einstein tensor. It is now easy to show that the Higgs term in the theory of the nematic when expressed in terms of the linearized Einstein tensors,

$$\begin{aligned} \sigma_{\mu\nu}\frac{1}{\partial^2}\sigma_{\mu\nu} &= G_{\mu\nu}\frac{1}{\partial^2}G_{\mu\nu} \\ &\to R, \end{aligned} \tag{6.23}$$

actually reduces to the Ricci tensor R, demonstrating that one recovers the Einstein–Hilbert action at distances large compared to the Higgs scale. Once more, this only holds in the linearized theory. This works in the same way in 3+1 (and higher) dimensions which is the easy way to demonstrate that gravitons have to be present [102, 103]. At least the linearized version of the Einstein–Hilbert action appears to be precisely coincident with the effective field theory describing the collective behaviour of the quantum nematic!

Although this all looks convincing there is still a gap in the conceptual understanding of what has happened with the geometry of the crystal in the presence of condensed dislocations. The emergence of gravity requires that the original spacetime defined by the crystal has to become diffeomorphic.

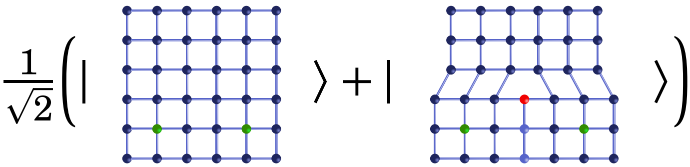


Figure 6.4: In the dislocation condensate (quantum nematic), the distance between two points (green dots) is in a coherent superposition of having zero, one or any number of half-line insertions (light blue) or dislocations (red dot) in between them, and therefore the number of lattice spacings in between them is undefined. This is equivalent to having the Einstein translations fully gauged: there is a diffeomorphism between configurations with any number of lattice spacings in between the two points.

The fields as of relevance to the dynamics of the nematic are healthy in this regard but they belong to the dual side. The analogy with the Mott insulator is now helpful: to demonstrate that gravity has emerged requires the demonstration that the spacetime of the original crystal is diffeomorphic and that is equivalent to demonstrating that in the vortex condensate the superfluid phase acquires a compact $U(1)$ gauge invariance. The diffeomorphic nature of the stress gauge fields telling about the excitations of the quantum nematic has in turn the same status as the gauge fields that render the vortex condensate to be a superconductor.

The good news is that we can use the same "first quantization" trick that helped us to understand the emergence of the stay-at-home gauge in the vortex condensate to close this conceptual gap. As for the vortices, it is easy to picture what happens to the metric of the crystal when the coherent superpositions of dislocation configurations associated with the dual stress superconductor are present. Let us repeat the exercise at the end of the previous section (Fig. 6.1), by comparing how two points some distance apart communicate with each other, but now focusing on the metric properties. This is illustrated in Fig. 6.4: imagine that no dislocation is present between the two points and one needs N jumps to get from one point to the other. However, this configuration is at energies less than the Higgs mass of the quantum nematic necessarily in coherent superposition with a configuration where a dislocation has moved through the line connecting the two points:

one now needs $N+1$ hops and since these configurations are in coherent superposition "$N = N+1$" and the geometry is now truly diffeomorphic!

However, there is one last caveat. Although translational symmetry is restored in the quantum nematic, the rotations are still in a fixed frame and even spontaneously broken! This is different from full Einstein gravity: in real spacetime also the Lorentz transformations (rotations in our Euclidean setting) are fully gauged. In order to understand this point, let us start from special relativity, which has the global symmetry of the Poincaré group comprising translations and Lorentz transformations. The translations form a subgroup, such that translational and rotational symmetry are easily distinguishable. More precisely, the generators of translations are ordinary derivatives ∂_μ which commute $[\partial_\mu, \partial_\nu] = 0$. In many ways, going from special to general relativity is from going from global to local Poincaré symmetry [115]. Indeed, referring to elasticity language, it seems to make sense to restore first translational and then rotational symmetry, ending up in a perfectly locally symmetric "liquid" state.

However, it has long been known that such "gauging of spacetime symmetry" is very intricate, which has to do with the definition of locality under such transformations. What happens is that local coordinate transformations of the form $x_\mu \to \xi_\mu(x_\nu)$, which are in fact local translations, can also correspond to local rotations. The local translations no longer form a subgroup, as the generators of translations should be augmented to those of *parallel translations*, defined by [116],

$$D_\mu = \partial_\mu + \Gamma_\mu{}^{\kappa\lambda} f_{\kappa\lambda}, \tag{6.24}$$

where $\Gamma_\mu{}^{\kappa\lambda}$ is the connection and $f_{\kappa\lambda}$ is the generator of local rotations. Such modified derivatives do not commute, and two consecutive translations may result in a finite rotation. Such symmetry structure is actually at the heart at everything non-linear happening in Einstein theory including black holes.

Going back to what we now know of the quantum nematic, it is clear that it cannot correspond to full GR, since rotational symmetry as reflected by disclinations is still gapped. Nevertheless, the identification between quantum nematics and *linearized* gravity is in perfect shape. Linearized gravity is a special and somewhat pathological limit of full GR, as it only applies to nearly globally Lorentz symmetric systems. It was quite some time ago realized that such systems are symmetric under global Lorentz transformations and infinitesimal coordinate transformations (see ch. 18,35 in Ref. [117]).

This is equivalent to fixing the Lorentz frame globally yet allow for infinitesimal Einstein translations. Under such conditions the equations of motion of linearized gravity follow automatically.

Here we have demonstrated that linearized gravity—a very peculiar limiting case of GR—is actually literally realized in a quantum nematic. The deeper reason is that in a quantum nematic the rotational symmetry of (Euclidean) spacetime is global and even spontaneously broken, while the restoration of the translational symmetry by the dislocation condensate has caused the fixed frame internal coordinate system of the crystal to turn into a geometry that is characterized by a covariance exclusively associated with infinitesimal translational coordinate transformations.

6.3 Summary and outlook

In so far as vortex duality is concerned we have presented here no more than a clarification. Living on the "dual side", where the Bose-Mott insulator appears as just a relativistic superconductor formed from vortices, the emerging stay-at-home local charge conservation from the canonical representation in terms of the Mott insulating phase of the Bose-Hubbard model is not manifestly recognizable. However, the dual vortex language contains all the information required to reconstruct precisely the nature of the field configurations of the "original" superfluid phase fields which are realized in the vortex superconductor. By inspecting these we identified a very simple but intriguing principle. The local charge conservation of the Mott insulator, associated with the emergent stay-at-home compact $U(1)$ gauge symmetry, is generated in the vortex condensate by the quantum mechanical principle that states in coherent superposition "are equally true at the same time"—the Schrödinger cat motive.

We find this simple insight useful since it yields a somewhat more general view on the nature of strong/weak dualities. We already emphasized that Mott insulators as defined through the local conservation of charge do not necessarily need a lattice. One does not have to dig deep to find an example: our dual superconductor is just a relativistic superconductor in 2+1d, which is in turn dual to a Coulomb phase that can also be seen as a superfluid. The charge associated with this superfluid is locally conserved in the superconductor, regardless of whether the superconductor lives on a lattice or in the continuum.

We find the emergent gauging of translational symmetry realized in the quantum nematic an even better example of the usefulness of this insight. Earlier work indicated that the relativistic version of this nematic is somehow associated with emergent gravity. Resting on the "coherent superposition" argument it becomes directly transparent what causes the gauging of the crystal coordinates: the condensed dislocations "shake the coordinates coherently" such that infinitesimal Einstein translations appear while the Lorentz frame stays fixed. This emergent symmetry imposes that the collective excitations of the quantum nematic have to be in one-to-one correspondence with linearized gravity.

Our message is that we have identified a mechanism for the "dynamical generation" of gauge symmetry which is very simple but also intriguing viewed from a general physics perspective: the quantum mechanics principle of states in coherent superposition being "equally true at the same time" translates directly to the principle that the global symmetry that is broken in the ordered state is turned into a gauge symmetry on the disordered side just by the quantum undeterminedness of the topological excitations in the dual condensate. This raises the interesting question: is quantum coherence required for the emergence of local symmetry, or can it also occur in classical systems?

This question relates directly to the spectacular recent discovery of "Dirac monopoles" in spin ice [118]. Castelnovo, Moessner and Sondhi [119] realized that the manifold of ground states ("frustration volume") of this classical geometrically frustrated spin problem is coincident with the gauge volume of a compact $U(1)$ gauge theory, with the ramification that it carries Dirac monopoles as topological excitations. All along it has been subject of debate to what extent these monopoles can be viewed as literal Dirac monopoles in the special "vacuum" realized in the spin ice, or rather half-bred cartoon versions of the real thing. With our recipe at hand it is obvious how to make them completely real: imagine the classical spin ice to fill up Euclidean spacetime, and after Wick rotation our "coherent superposition principle" would have turned the frustration volume of the classical problem into a genuine gauge volume since by quantum superposition all degenerate states would be "equally true at the same time".

The ambiguity associated with the classical spin ice monopoles is rooted in the role of time. In principle, by doing time-resolved measurements one can observe every particular state in the frustration volume and this renders these states to be not gauge equivalent. However, all experiments which have revealed the monopoles involved large, macroscopic time scales. One can pose the question whether it is actually possible under these conditions to define observables that can discriminate between the “fake” monopoles of spin ice and the monopoles of Dirac. Perhaps the answer is pragmatic: as long as ergodicity is in charge, one can rely on the ensemble average instead of the time average, and as long as the time scale of the experiment is long enough such that one is in the ergodic regime, the frustration volume will “disappear” in the ensemble average. For all practical purposes one is then dealing with a genuinely emergent gauge symmetry which tells us that in every regard the spin ice monopole is indistinguishable from the Dirac monopole.

Chapter 7

Conclusions

Now that I have presented all results in detail, it is time to review to where this has led us. First, let us reflect on the obtained results.

7.1 Summary of results

In one sentence each, the main chapters can be summed up as follows:

(3) The vortex-unbinding transition causes the demise of current conservation, and is a valid description for systems of any dimension larger than 2;

(4) All electrodynamic phenomena related to Abrikosov vortices are comprised in a single equation for the vortex world sheet;

(5) There is a new state of matter, called type-II Mott insulator, which features quantized vortices of electric current, that is directly accessible in experiment;

(6) The gauge symmetry due to an emerging local conservation law corresponds to a superposition of all possible configurations of topological defects.

7.2 Outlook

These quantitative and qualitative outcomes are of course the main results of this thesis. However, they also provoke thought on some of the deeper principles surrounding ordered systems, gauge symmetry and the universality of vortex-unbinding transitions.

7.2.1 The Landau paradigm

As mentioned in §§2.1,2.2.1, ever since Landau employed an *order parameter* to describe the formation of the superfluid, this has been the prevailing *modus operandi* in the theory of ordered states and (continuous) phase transitions. This sometimes goes under the name of the Landau–Ginzburg–Wilson paradigm. The most important property of the order parameter is that it is a local variable, namely a function on every point in space.

But starting with the Kosterlitz–Thouless transition, several phase transitions have been identified that seem to fall outside of this characterization. Thus, the KT transition is sometimes said to be a phase transition without spontaneous symmetry breaking, or a phase transition of infinite order (instead of second order). It is also often claimed that there is no order parameter for the phase across the transition. The theme of a different kind of order really caught on with the advent of the quantum Hall effect. The distinct quantum Hall states are characterized by a quantum number called the Chern number, which is *topological*, meaning it can only be defined for the system as a whole. From it has evolved the study of topological order, with Xiao-Gang Wen as one of the major pioneers [120]. He argues that also the concept of symmetry groups is too restricted to capture all phase transitions and should be extended to *projective symmetry groups* [121].

To distinguish one ordered state from another it is necessary to define some quantity which will differ for distinct ordered states. The Landau order parameter fulfils this task, but because it is local, it seems not universal enough to capture for instance topological order. It is often said that "topological order is beyond the Landau paradigm". However throughout this thesis and emphasized in chapter 6, we have seen that what is local in one description is extremely non-local in the dual description. For example, the Mott insulator is a vortex condensate, where the density of the vortex liquid is the order parameter that obtains an expectation value, and the associated phase variable is broken spontaneously. From the phase variable point of view, all correlations are lost, but in dual language it is just condensation of a local order parameter. As such, it may well be that the topological orders for which currently no local order parameter can be defined, will be made to do so by a suitable duality transformation. If this turns out to be achievable, then possibly the Landau paradigm will hold up and the intuition-pleasing notion of a local order parameter can survive.

This is also the attitude taken in Hopf symmetry breaking [20–26]. A Hopf algebra or quantum group is a mathematical generalization of an ordinary symmetry group, that treats topological defects and particle excitations on equal footing. Both symmetry breaking and symmetry restoration by defect condensation are contained within this formalism. This should be fertile ground for further explorations of these matters.

7.2.2 Quantum liquid crystals

As mentioned in the introduction §1.2, the original topic of this thesis was to be the quantum liquid crystals. Next to the correspondence between quantum nematics and linearized gravity (§6.2), electronic quantum liquid crystals have received much attention lately, because they seem to be present in the pseudogap phase of underdoped cuprates (§5.1.2). Therefore, they make up an interesting and relevant topic in its own right.

The difference between classical and quantum liquid crystals is that the latter can have superpositions of Burgers vector orientations (§6.2.1) as defect condensates. Therefore it admits many more ground states than the classical varieties. This leaves room for surprises, and it would be very useful to have a complete mathematical classification of all possible quantum condensates, analogous to the group symmetry scheme for regular ordered states. There are 17 so-called wallpaper groups of infinite tilings of the spatial plane. These would correspond to all possible two-dimensional crystals (quantum crystals are 2+1 dimensional and do not suffer from the Mermin–Wagner theorem). Melting such crystals by dislocation proliferation leads to the corresponding quantum liquid crystals.

The Hopf algebra or quantum group formalism advertises its ability to perform this classification [23–25]. It is well suited to handle the non-commutativity present in the marriage between translations and rotations, and also automatically provides the classification of topological defects in each liquid crystal phase. The downside is that there is no recipe to list all possible inequivalent quantum superposition condensates, so they must be guessed and written out by hand. There may be also some mathematical intricacies related to the infiniteness of the lattice groups (cf. Ref. [122]).

Carrying this out for all condensates of all 17 groups seems a daunting task. My advice would be to start with the square and triangular lattice (p4m and p6m in crystallography language) which seem most relevant to experiment.

7.2.3 Vortex duality and fermions

All of this work so far has concerned bosonic physics only. Even the superconductor was treated exclusively as a Bose condensate of Cooper pairs. The main reason is that fermions are in fact much harder to describe theoretically, all of it related to their minus signs (see e.g. Ref. [123]). Still, given the very general considerations presented here, and the relation between disorder in the real and order in the dual variables stressed in chapter 6, one cannot help but think that the defect-mediated melting should prevail also in systems other than purely bosonic.

There are several ways that may indeed establish vortex condensation in fermionic physics. One way could be to somehow to impose the Pauli-exclusion interactions as additional constraints in the path integral. This had led to some interesting insights into quantum criticality [124], but does not reveal how to continue this line of thought. Another possibility may be to let the vortices in a bosonic (phase) field take care of the phase transition, and wire in the fermionic physics by a separate particle species. This is the approach take in the many slave-particle models available on the market, see e.g. Ref. [58]. Fascinatingly, a very recent work literally employs the vortex duality in the *fermionic* Mott insulator by mode expansion, where only the $k = 0$ mode relates to the vortex condensate and all finite-momentum modes concern the fermionic spin dynamics [94]. It would also be interesting to see whether such principles may be applied to the topological Mott insulators that are in vogue these days [125, 126].

7.2.4 Quantum vs. classical

We conclude with perhaps the most fundamental question in the realm of condensed matter physics: Where is the border between classical and quantum phenomena, if it exists at all?

It is certainly not true that quantum mechanics only manifests is itself at energies $\hbar/\tau > k_{\mathrm{B}}T$, where τ is some characteristic time scale. Not only is for instance light an inherently quantum mechanical wave/particle, but all materials in everyday life only exist because of quantum mechanics. What I mean is that it is not enough to have atoms + Coulomb interaction + Pauli exclusion. A crystal is rigid because of spontaneous symmetry breaking, leading to long-range correlations communicated by Goldstone modes. Even disordered systems such as glasses are dominated by this effect on other length scales. Only truly simple liquids such as liquid nitrogen seem to fall outside of this world, and even there Van der Waals interactions are quantum mechanical in origin.

In fact, a superconductor, often referred to as a macroscopic quantum system, is not any different in this respect from a crystal. It is just an internal instead of an external (spacetime) symmetry that is spontaneously broken. Again I take the point of view that *any* ordered system must be some sort of condensate, and as such must be distinguishable from a completely unordered system, presumably by a symmetry property. Thus a superconductor is at the basic level as classical as a crystal, or a crystal is as quantum mechanical as a superconductor, whichever one prefers. The Goldstone mode conveys the rigidity of the condensate. Spontaneous symmetry breaking is caused by coupling to an enormous amount of just-above-zero-energy states called the thin spectrum, and is therefore unavoidable in many-body physics (see e.g. [127, 128]).

We touched upon this theme in the very last paragraph of §6.3: can we distinguish the 'magnetic monopole' excitations in spin ice from true Dirac monopoles? In fact, the experiments available at the moment cannot resolve the difference between time-averaged and ensemble-averaged results, and therefore see only the 'classical' consequences of the monopoles. This once more suggests that the specialities of quantum mechanics lie in the role of time. In the condensates this is not noticeable. We may conjecture that the true dividing line between classical media, including superconductors, and quantum stuff would manifest itself in some way by the role of time. Perhaps whenever Wick rotation to imaginary time is possible, the 'true' quantum features are obscured, or irrelevant for the final outcome.

Surely, in high-energy particle physics Wick rotation is part of the standard toolbox, and those processes are definitely purely quantum mechanical. Or are they not? One could argue that the calculation of the outcomes of a scattering process from the infinite past to the infinite future is just as classical a result as the propagation sound waves in a crystal, in the sense that the special properties of time play no role. If this is so—and I cannot at all claim to have in any way established or even corroborated it—then only things like anomalies are 'truly' quantum.

The classicalness of all condensates also underlies much confusion surrounding the quantum measurement problem or "collapse of the wave function". Nobody disputes that the statistical interpretation of quantum mechanics, where the probability amplitudes are meaningful in many repetitions of the same experiment, is extremely accurate. This is the ensemble average. But what happens for each individual experiment and whether quantum mechanics has anything to say about this, is still unclear. The problem arises in how to couple a single particle like a free electron to a condensate such as a photographic plate. The ground state wave function of the condensate is as classical as can be, and not only does the quantum information of the electron get lost in the huge amount of degrees of freedom, the fact that spontaneous symmetry breaking has already taken place prevents a straightforward coupling of the quantum particle to the classical system. A very interesting proposal is that gravity prevents the quantum superposition of a heavy enough object by the mismatch of deformed spacetimes [129]. Moreover, the deeper mechanism behind this seems to be the special role of time [130].

It is my opinion that the mysteries of the peculiar role of time are largely unresolved. It has not only bearing on things like quantum gravity, but definitely also on the divide between quantumness and classicalness in the everyday world around us.

References

[1] H. A. Kramers and G. H. Wannier. *Statistics of the Two-Dimensional Ferromagnet. Part I.* Physical Review, **60** (3):252–262, Aug 1941

[2] R. Savit. *Duality in field theory and statistical systems*. Reviews of Modern Physics, **52** (2):453–487, Apr 1980

[3] J. B. Kogut. *An introduction to lattice gauge theory and spin systems*. Reviews of Modern Physics, **51** (4):659–713, Oct 1979

[4] F. J. Wegner. *Duality in Generalized Ising Models and Phase Transitions without Local Order Parameters*. Journal of Mathematical Physics, **12** (10):2259, 1971

[5] Y. Nambu. *Quasi-Particles and Gauge Invariance in the Theory of Superconductivity*. Physical Review, **117** (3):648–663, Feb 1960

[6] J. Goldstone. *Field theories with " Superconductor " solutions*. Il Nuovo Cimento, **19**:154–164, 1961

[7] J. Goldstone, A. Salam and S. Weinberg. *Broken Symmetries*. Physical Review, **127**:965–970, 1962

[8] N. D. Mermin and H. Wagner. *Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models*. Physical Review Letters, **17** (22):1133–1136, Nov 1966

[9] P. C. Hohenberg. *Existence of Long-Range Order in One and Two Dimensions*. Physical Review, **158** (2):383–386, Jun 1967

[10] S. Coleman. *There are no Goldstone bosons in two dimensions*. Communications in Mathetical Physics, **31** (4):259–264, 1973

[11] J. M. Kosterlitz and D. J. Thouless. *Ordering, metastability and phase transitions in two-dimensional systems*. Journal of Physics C: Solid State Physics, **6** (7):1181, 1973

[12] V. L. Berenzinskii. Soviet Journal of Experimental and Theoretical Physics, **32**:493, 1970

[13] D. R. Nelson and B. I. Halperin. *Dislocation-mediated melting in two dimensions*. Physical Review, **B19** (5):2457–2484, Mar 1979

[14] A. P. Young. *Melting and the vector Coulomb gas in two dimensions*. Physical Review, **B19** (4):1855–1866, Feb 1979

[15] P. W. Anderson. *Plasmons, Gauge Invariance, and Mass*. Physical Review, **130** (1):439–442, Apr 1963

[16] S. Elitzur. *Impossibility of spontaneously breaking local symmetries*. Physical Review, **D12** (12):3978–3982, Dec 1975

[17] S. Sachdev. *Quantum phase transitions* (Cambridge University Press, 1999)

[18] R. P. Feynman and A. R. Hibbs. *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965)

[19] G. D. Mahan. *Many-Particle Physics* (Plenum, New York, N.Y., 1993), 2nd edition

[20] M. Wild Propitius and F. A. Bais. *Discrete Gauge Theories*. In G. Semenoff and L. Vinet, eds., *Particles and Fields*, CRM Series in Mathematical Physics, pages 353–439 (Springer–Verlag, New York, NY, 1998). `arXiv:hep-th/9511201`

[21] F. A. Bais, B. J. Schroers and J. K. Slingerland. *Hopf symmetry breaking and confinement in (2+1)-dimensional gauge theory*. Journal of High Energy Physics, **2003** (05):068, 2003. `arXiv:hep-th/0205114`

[22] J. K. Slingerland. *Hopf Symmetry and its breaking; Braid Statistics and Confinement in Planar Physics*. Ph.D. thesis, University of Amsterdam, 2002

[23] F. A. Bais and C. J. M. Mathy. *Defect-mediated melting and the breaking of quantum double symmetries*. Physical Review, **B73** (22):224120, Jun 2006. `arXiv:cond-mat/0602101`

[24] F. A. Bais and C. J. M. Mathy. *The breaking of quantum double symmetries by defect condensation*. Annals of Physics, **322** (3):552–598, 2007. `arXiv:cond-mat/0602115`

[25] C. J. M. Mathy and F. A. Bais. *Nematic phases and the breaking of double symmetries*. Annals of Physics, **322**:709–735, 2007. `arXiv:cond-mat/0602109`

[26] A. J. Beekman. *Quantum double symmetries of the even dihedral groups and their breaking*. Master's thesis, University of Amsterdam, 2005

[27] E. Fradkin and S. H. Shenker. *Phase diagrams of lattice gauge theories with Higgs fields*. Physical Review, **D19** (12):3682–3697, Jun 1979

[28] H. Kleinert. *Gauge Fields in Condensed Matter, Vol.I Superflow and Vortex Lines* (World Scientific, Singapore, 1989). ISBN 9971-50-210-9 `http://users.physik.fu-berlin.de/~kleinert/kleinert/?p=booklist&details=1`

[29] M. P. A. Fisher and D. H. Lee. *Correspondence between two-dimensional bosons and a bulk superconductor in a magnetic field*. Physical Review, **B39** (4):2756–2759, Feb 1989

[30] I. F. Herbut and Z. Tešanović. *Critical Fluctuations in Superconductors and the Magnetic Field Penetration Depth*. Physical Review Letters, **76** (24):4588–4591, Jun 1996. `arXiv:cond-mat/9605185`

[31] V. Cvetković and J. Zaanen. *Vortex duality: Observing the dual nature using order propagators*. Physical Review, **B74**:134504, 2006. `arXiv:cond-mat/0511586`

[32] A. K. Nguyen and A. Sudbø. *Topological phase fluctuations, amplitude fluctuations, and criticality in extreme type-II superconductors*. Physical Review, **B60** (22):15307–15331, Dec 1999. `arXiv:cond-mat/9907385`

[33] J. Hove and A. Sudbø. *Anomalous Scaling Dimensions and Stable Charged Fixed Point of Type-II Superconductors*. Physical Review Letters, **84** (15):3426–3429, Apr 2000. arXiv:cond-mat/0002197

[34] N. Nagaosa and P. A. Lee. *Confinement and Bose condensation in gauge theory of high-T_c superconductors*. Physical Review, **B61**:9166–9175, Apr 2000

[35] A. Zee. *Some general remarks on superconductivity*. Physica, **A281** (1-4):442–449, 2000

[36] M. P. A. Fisher. *Duality in low dimensional quantum field theories*. In *Strong interactions in low dimensions*, Physics and Chemistry of Materials with Low-Dimensional Structures, pages 419–438 (Springer, 2004)

[37] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992). ISBN 0-0691-08775-X

[38] M. Kaku and K. Kikkawa. *Field theory of relativistic strings. I. Trees*. Physical Review, **D10** (4):1110–1133, Aug 1974

[39] E. Witten. *Non-commutative geometry and string field theory*. Nuclear Physics, **B268** (2):253–294, 1986

[40] V. Cvetković. *Quantum Liquid Crystals*. Ph.D. thesis, Leiden University, March 2006. https://openaccess.leidenuniv.nl/handle/1887/4456

[41] H. Kleinert. *Gauge Fields in Condensed Matter, Vol.II Stress and Defects* (World Scientific, Singapore, 1989). ISBN 9971-50-210-9 http://users.physik.fu-berlin.de/~kleinert/kleinert/?p=booklist&details=2

[42] H. Kleinert. *Mulivalued Fields in Condensed Matter, Electromagnetism, and Gravitation* (World Scientific, Singapore, 2008). ISBN 978-981-279-170-2

[43] H. Kleinert and J. Zaanen. *Nematic world crystal model of gravity explaining absence of torsion in spacetime*. Physics Letters, **A324** (5-6):361–365, 2004. arXiv:cond-mat/0309379

[44] M. Pitkin, S. Reid, S. Rowan and J. Hough. *Gravitational Wave Detection by Interferometry (Ground and Space)*. Living Reviews in Relativity, **14** (5), 2011
http://www.livingreviews.org/lrr-2011-5

[45] V. Cvetković and J. Zaanen. *Quantum Smectic as a Dislocation Higgs Phase*. Physical Review Letters, **97**:045701, 2006. arXiv:cond-mat/0604639

[46] M. Franz. *Vortex-boson duality in four space-time dimension*. Euro Physics Letters, **77**:47005, 2007. arXiv:cond-mat/0607310

[47] S.-J. Rey. *Higgs mechanism for Kalb–Ramond gauge field*. Physical Review, **D40**:3396–3401, 1989

[48] C. Marshall and P. Ramond. *Field theory of the interacting string: the closed string*. Nuclear Physics, **B85**:375–414, 1975

[49] E. di Grezia and S. Esposito. *Minimal coupling of the Kalb–Ramond field to a scalar field*. International Journal of Theoretical Physics, **43** (2):445–456, Feb 2004

[50] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch and I. Bloch. *Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms*. Nature, **415** (6867):39–44, 2002

[51] M. Tinkham. *Introduction to superconductivity* (McGraw-Hill, New York, 1996), 2nd edition. ISBN 0-07-064878-6

[52] A. A. Abrikosov. *On the magnetic properties of superconductors of the second group*. Soviet Journal of Experimental and Theoretical Physics, **5**:1174, 1957

[53] N. D. Mermin. *The topological theory of defects in ordered media*. Reviews of Modern Physics, **51**:591–648, 1979

[54] M. P. A. Fisher, P. B. Weichman, G. Grinstein and D. S. Fisher. *Boson localization and the superfluid–insulator transition*. Physical Review, **B40** (1):546–570, Jul 1989

[55] R. Fazio and H. van der Zant. *Quantum phase transitions and vortex dynamics in superconducting networks*. Physics Reports, **355** (4):235–334, 2001. arXiv:cond-mat/0011152

[56] C. Bruder, R. Fazio and G. Schön. *The Bose-Hubbard model: from Josephson junction arrays to optical lattices*. Annalen der Physik, **14** (9):566–577, 2005

[57] I. Affleck and J. B. Marston. *Large-n limit of the Heisenberg-Hubbard model: Implications for high-Tc superconductors*. Physical Review, **B37** (7):3774–3777, Mar 1988

[58] P. A. Lee, N. Nagaosa and X.-G. Wen. *Doping a Mott insulator: Physics of high-temperature superconductivity*. Reviews of Modern Physics, **78**:17–85, 2006. `arXiv:cond-mat/0410445`

[59] S. D. Huber, B. Theiler, E. Altman and G. Blatter. *Amplitude Mode in the Quantum Phase Model*. Phys. Rev. Lett., **100**:050404, Feb 2008. `arXiv:0709.4436`

[60] H. B. Nielsen and P. Olesen. *Vortex-line models for dual strings*. Nuclear Physics, **B61**:45–61, 1973

[61] A. M. Polyakov. *Gauge fields and strings* (Harwood Academic Publishers, 1987)

[62] J. Schwinger. *Particles, Sources and Fields* (Addison-Wesley, Reading, MA, 1970)

[63] E. di Grezia, S. Esposito and A. Naddeo. *Quantum phase excitations in Ginzburg–Landau superconductors*. International Journal of Modern Physics, **B20**:737–745, 2006. `arXiv:hep-th/0411144`

[64] J. Badur. *Generalized Helmholtz theorem for multi-forms*. International journal of engineering science, **27**:943–953, 1989

[65] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani and M. K. Oberthaler. *Direct Observation of Tunneling and Nonlinear Self-Trapping in a Single Bosonic Josephson Junction*. Physical Review Letters, **95** (1):010402, Jun 2005

[66] S. Levy, E. Lahoud, I. Shomroni and J. Steinhauer. *The a.c. and d.c. Josephson effects in a Bose-Einstein condensate*. Nature, **449**:579–583, 10 2007

[67] H. Rogalla and P. H. Kes, eds. *100 Years of Superconductivity* (CRC Press, New York, 2011)

[68] D. R. Nelson. *Vortex Entanglement in High-T_c Superconductors*. Physical Review Letters, **60** (19):1973–1976, May 1988

[69] B. Rosenstein and D. Li. *Ginzburg–Landau theory of type II superconductors in magnetic field*. Reviews of Modern Physics, **82** (1):109–168, Jan 2010

[70] G. Blatter, M. V. Feigel'man, V. B. Geshkenbein, A. I. Larkin and V. M. Vinokur. *Vortices in high-temperature superconductors*. Reviews of Modern Physics, **66** (4):1125–1388, Oct 1994

[71] L. N. Bulaevskii and E. M. Chudnovsky. *Electromagnetic Radiation from Vortex Flow in Type-II Superconductors*. Physical Review Letters, **97** (19):197002, Nov 2006

[72] L. N. Bulaevskii and A. E. Koshelev. *Radiation from a Single Josephson Junction into Free Space due to Josephson Oscillations*. Physical Review Letters, **97** (26):267001, Dec 2006

[73] L. D. Landau and E. M. Lifshitz. *Electrodynamics of continuous media*. Number 8 in Course of theoretical physics (Pergamon, Oxford, 1960). ISBN 0080091059

[74] B. D. Josephson. *Potential differences in the mixed state of type II superconductors*. Physics Letters, **16** (3):242 – 243, 1965

[75] S. Savel'ev, V. A. Yampol'skii, A. L. Rakhmanov and F. Nori. *Terahertz Josephson plasma waves in layered superconductors: spectrum, generation, nonlinear and quantum phenomena*. Reports on Progress in Physics, **73** (2):026501, 2010. `arXiv:0903.2969`

[76] R. G. Mints and I. B. Snapiro. *Electromagnetic waves in a Josephson junction in a thin film*. Physical Review, **B51** (5):3054–3057, Feb 1995

[77] K. F. Warnick and P. Russer. *Two, Three and Four-Dimensional Electromagnetics Using Differential Forms*. Turkish Journal of Electrical Engineering & Computer Sciences, **14** (1):153–172, 2006

[78] M. Vojta. *Lattice symmetry breaking in cuprate superconductors: stripes, nematics, and superconductivity*. Advances in Physics, **58** (6):699–820, 2009

[79] J. Zaanen and O. Gunnarsson. *Charged magnetic domain lines and the magnetism of high-T_c oxides*. Physical Review, **B40** (10):7391–7394, Oct 1989

[80] S. A. Kivelson, I. P. Bindloss, E. Fradkin, V. Oganesyan, J. M. Tranquada, A. Kapitulnik and C. Howald. *How to detect fluctuating stripes in the high-temperature superconductors*. Reviews of Modern Physics, **75** (4):1201–1241, Oct 2003

[81] C. M. Varma. *Theory of the pseudogap state of the cuprates*. Physical Review, **B73** (15):155113, Apr 2006

[82] S. A. Kivelson, E. Fradkin and V. J. Emery. *Electronic liquid-crystal phases of a doped Mott insulator*. Nature, **393**:550–553, 1998. `arXiv:cond-mat/9707327`

[83] V. Cvetkovic, Z. Nussinov, S. Mukhin and J. Zaanen. *Observing the fluctuating stripes in high-T_c superconductors*. Euro Physics Letters, **81** (2):27001, 2008. `arXiv:cond-mat/0607402`

[84] A. Mesaros, K. Fujita, H. Eisaki, S. Uchida, J. C. Davis, S. Sachdev, J. Zaanen, M. J. Lawler and E.-A. Kim. *Topological Defects Coupling Smectic Modulations to Intra–Unit-Cell Nematicity in Cuprates*. Science, **333** (6041):426–430, 2011. `arXiv:1108.0487`

[85] V. J. Emery and S. A. Kivelson. *Importance of phase fluctuations in superconductors with small superfluid density*. Nature, **374**:434–437, 03 1995

[86] M. Franz. *Superconductivity: Importance of fluctuations*. Nature Physics, **3**:686–687, 2007

[87] M.-C. Cha and J.-W. Lee. *Finite-Temperature Phase Transitions in a Two-Dimensional Boson Hubbard Model*. Phys. Rev. Lett., **98**:266406, Jun 2007. `arXiv:0704.0396`

[88] T. P. Polak and T. K. Kopeć. *Finite-temperature effects on the superfluid Bose–Einstein condensation of confined ultracold atoms in three-dimensional optical lattices*. Journal of Physics B: Atomic, Molecular and Optical Physics, **42** (9):095302, 2009. arXiv:1006.0357

[89] K. A. Delin and A. W. Kleinsasser. *Stationary properties of high-critical-temperature proximity effect Josephson junctions*. Superconductor Science and Technology, **9** (4):227, 1996

[90] I. Bozovic, G. Logvenov, M. A. J. Verhoeven, P. Caputo, E. Goldobin and M. R. Beasley. *Giant Proximity Effect in Cuprate Superconductors*. Physical Review Letters, **93** (15):157002, Oct 2004

[91] D. Marchand, L. Covaci, M. Berciu and M. Franz. *Giant Proximity Effect in a Phase-Fluctuating Superconductor*. Phys. Rev. Lett., **101**:097004, Aug 2008

[92] P. Phillips. *Colloquium: Identifying the propagating charge modes in doped Mott insulators*. Reviews of Modern Physics, **82** (2):1719–1742, May 2010. arXiv:1001.5270

[93] J. Zaanen. *A modern, but way too short history of the theory of superconductivity at a high temperature*. In H. Rochalla and P. H. Kes, eds., *100 years of superconductivity* (Chapman and Hall, 2011). arXiv:1012.5461

[94] D. F. Mross and T. Senthil. *Decohering the Fermi liquid: A dual approach to the Mott Transition*, 2011. arXiv:1107.4125

[95] E. Fradkin and S. A. Kivelson. *Liquid-crystal phases of quantum Hall systems*. Physical Review, **B59** (12):8065–8072, Mar 1999. arXiv:cond-mat/9810151

[96] V. Hinkov, D. Haug, B. Fauqué, P. Bourges, Y. Sidis, A. Ivanov, C. Bernhard, C. T. Lin and B. Keimer. *Electronic Liquid Crystal State in the High-Temperature Superconductor $YBa_2Cu_3O_{6.45}$*. Science, **319** (5863):597–600, 2008. arXiv:0807.1861

[97] R. Daou, J. Chang, D. LeBoeuf, O. Cyr-Choiniere, F. Laliberte, N. Doiron-Leyraud, B. J. Ramshaw, R. Liang, D. A. Bonn, W. N. Hardy and L. Taillefer. *Broken rotational symmetry in the pseudogap phase*

of a high-T_c superconductor. Nature, **463** (7280):519–522, Jan 2010. arXiv:0909.4430

[98] M. J. Lawler, K. Fujita, J. Lee, A. R. Schmidt, Y. Kohsaka, C. K. Kim, H. Eisaki, S. Uchida, J. C. Davis, J. P. Sethna and E.-A. Kim. *Intra-unit-cell electronic nematicity of the high-T_c copper-oxide pseudogap states*. Nature, **466** (7304):347–351, 2010. arXiv:1007.3216

[99] R. A. Borzi, S. A. Grigera, J. Farrell, R. S. Perry, S. J. S. Lister, S. L. Lee, D. A. Tennant, Y. Maeno and A. P. Mackenzie. *Formation of a Nematic Fluid at High Fields in $Sr_3Ru_2O_7$*. Science, **315** (5809):214–217, 2007. arXiv:cond-mat/0612599

[100] J.-H. Chu, J. G. Analytis, K. De Greve, P. L. McMahon, Z. Islam, Y. Yamamoto and I. R. Fisher. *In-Plane Resistivity Anisotropy in an Underdoped Iron Arsenide Superconductor*. Science, **329** (5993):824–826, 2010. arXiv:1002.3364

[101] J. Zaanen, Z. Nussinov and S. I. Mukhin. *Duality in 2+1D Quantum Elasticity: superconductivity and Quantum Nematic Order*. Annals of Physics, **310** (1):181–260, March 2004. arXiv:cond-mat/0309397

[102] Y. Zhu and Y. Jiang. *World crystal: extending to four dimensional*. General Relativity and Gravity, pages 1–7, 2011

[103] A. J. Beekman, D. Sadri and J. Zaanen. In preparation

[104] G. E. Volovik. *Superfluid analogies of cosmological phenomena*. Physics Reports, **351** (4):195 – 348, 2001. arXiv:gr-qc/0005091

[105] G. E. Volovik. *The Universe in a Helium Droplet* (Oxford University Press, USA, 2003). ISBN 978-0198507826

[106] G. E. Volovik. *Phenomenology of Effective Gravity*. In H. Arodz, J. Dziarmaga and W. H. Zurek, eds., *Patterns of Symmetry Breaking*, volume 127 of *NATO Science Series*, pages 381–404 (Springer Netherlands, 2004). ISBN 978-94-007-1029-0. arXiv:gr-qc/0304061

[107] M. Visser. *Acoustic black holes: horizons, ergospheres and Hawking radiation*. Classical and Quantum Gravity, **15** (6):1767, 1998. arXiv:gr-qc/9712010

[108] L. J. Garay, J. R. Anglin, J. I. Cirac and P. Zoller. *Sonic Analog of Gravitational Black Holes in Bose-Einstein Condensates*. Physical Review Letters, **85** (22):4643–4647, Nov 2000. `arXiv:gr-qc/0002015`

[109] G. Jannes. *Emergent gravity: the BEC paradigm*. Ph.D. thesis, Universidad Complutense de Madrid, 2009. `arXiv:0907.2839`

[110] G. Chapline. *Quantum Phase Transitions and the Failure of Classical General Relativity*. International Journal of Modern Physics, **A18**:3587–3590, 2003. `arXiv:gr-qc/0012094`

[111] A. Mesaros, D. Sadri and J. Zaanen. *Parallel transport of electrons in graphene parallels gravity*. Physical Review, **B82** (7):073405, Aug 2010. `arXiv:0909.2703`

[112] Z.-C. Gu and X.-G. Wen. *A lattice bosonic model as a quantum theory of gravity*. 2006. `arXiv:gr-qc/0606100`

[113] S.-C. Zhang and J. Hu. *A Four-Dimensional Generalization of the Quantum Hall Effect*. Science, **294** (5543):823–828, 2001. `arXiv:cond-mat/0110572`

[114] S.-F. Wu, B. Wang, X.-H. Ge and G.-H. Yang. *Deriving the gravitational field equation and horizon entropy for arbitrary diffeomorphism-invariant gravity from a spacetime solid*. Physical Review, **D81** (4):044010, Feb 2010. `arXiv:0909.1367`

[115] F. Gronwald and F. W. Hehl. *On the Gauge Aspects of Gravity*. 1996. `arXiv:gr-qc/9602013`

[116] F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester. *General relativity with spin and torsion: Foundations and prospects*. Reviews of Modern Physics, **48** (3):393–416, Jul 1976

[117] C. W. Misner, K. S. Thorne and J. A. Wheeler. *Gravitation*. Physics Series (W. H. Freeman, 1973). ISBN 0716703440

[118] S. T. Bramwell and M. J. P. Gingras. *Spin Ice State in Frustrated Magnetic Pyrochlore Materials*. Science, **294** (5546):1495–1501, 2001. `arXiv:cond-mat/0201427`

[119] C. Castelnovo, R. Moessner and S. L. Sondhi. *Magnetic monopoles in spin ice*. Nature, **451**:42–45, 01 2008. `arXiv:0710.5515`

[120] X.-G. Wen. *Quantum Field Theory of Many-Body Systems* (Oxford University Press, Oxford, 2004). ISBN 0198530943

[121] X. Wen. *Quantum orders and symmetric spin liquids*. Physical Review, **B65** (16):165113, 2002. `arXiv:cond-mat/0107071`

[122] T. H. Koornwinder, F. A. Bais and N. M. Muller. *Tensor Product Representations of the Quantum Double of a Compact Group*. Communications in Mathematical Physics, **198** (1):157–186, 1998. `arXiv:q-alg/9712042`

[123] J. Zaanen. *Quantum Critical Electron Systems: The Uncharted Sign Worlds*. Science, **319** (5867):1205–1207, 2008

[124] F. Krüger and J. Zaanen. *Fermionic quantum criticality and the fractal nodal surface*. Physical Review, **B78**:035104, Jul 2008

[125] S. Raghu, X.-L. Qi, C. Honerkamp and S.-C. Zhang. *Topological Mott Insulators*. Physical Review Letters, **100**:156401, Apr 2008

[126] D. Pesin and L. Balents. *Mott physics and band topology in materials with strong spin-orbit interaction*. Nature Physics, **6**:376–381, 2010

[127] J. Zaanen. *The Classical Condensates: from crystals to Fermi-liquids* (Leiden University, 1996). Lecture notes

[128] J. van Wezel and J. van den Brink. *Spontaneous symmetry breaking and decoherence in superconductors*. Physical Review, **B77**:064523, Feb 2008

[129] J. van Wezel and T. H. Oosterkamp. *A nanoscale experiment measuring gravity's role in breaking the unitarity of quantum dynamics*. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, 2011. `arXiv:0912.3675`

[130] J. van Wezel. *Broken Time Translation Symmetry as a Model for Quantum State Reduction*. Symmetry, **2** (2):582–608, 2010. `arXiv:0912.4202`

Samenvatting

Om de verschijnselen in de natuur te beschrijven, hebben we niet alleen de fundamentele wetten—zoals de Coulombkracht van elektromagnetische wisselwerking—nodig, maar ook een manier om uit te leggen hoe deeltjes zich onder die wetten collectief gedragen. Het blijkt dat elementaire deeltjes *volledig* identiek zijn, dus dat we bijvoorbeeld het ene elektron op geen enkele manier van een andere kunnen onderscheiden. Daarnaast zijn er buitengewoon veel van die identieke deeltjes, een gram water bestaat uit grofweg 10^{23} moleculen. Als we dus willen weten hoe de materie om ons heen: vloeistoffen, tafels, koperdraden of neutronensterren zich gedragen, dan zijn we in de gelukkige toestand dat we maar een paar simpele ingrediënten hebben, in gigantische hoeveelheden, waardoor alle onbelangrijke effecten vrijwel altijd uitmiddelen. Inderdaad is het mogelijk om een natuurkundige vergelijking op te schrijven die in principe alle deeltjes en hun wisselwerkingen bevat, en dus naïef gezien alle antwoorden in zich moet dragen. Maar aangezien een systeem van al slechts drie deeltjes op dit moment wiskundig onoplosbaar is, zou die aanpak snel vruchteloos en onmogelijk worden.

Collectief gedrag

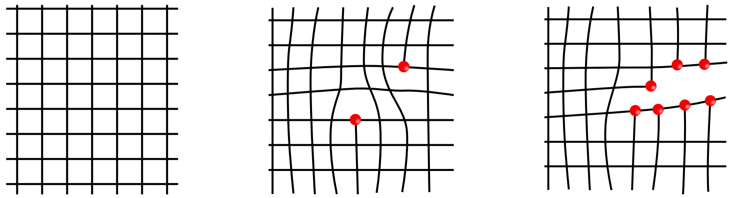
Het is ook overbodig, wat met een allegorie goed in te zien is. Als ik een mayonaise-uitvinder ben, is het handig om te weten hoe emulsies precies werken, en dat lecithine in het eigeel een goede olie/water-emulgator is, maar cholesterol een water/olie-emulgator, en dat de toevoeging van mosterd daardoor schier onontbeerlijk is. De kok echter, wil gewoon het recept hebben, en de mayonaise produceren. De restauranthouder wil dat de kok lekkere gerechten, al dan niet met mayonaise, maakt, de eigenaar wil enkel een kwalitatief goede en efficiënte bedrijfsvoering, zich niet bekommerend om welke ingrediënten dan ook. De gast verlangt simpelweg een smakelijke maaltijd. Kortom, alhoewel de geserveerde maaltijden mayonaise bevatten, is de mate waarin dat naar voren komt in de overkoepelende beschrijving afhankelijk van het niveau waarop je het beschouwt. Op een hoog niveau willen we alleen het resulterende, collectieve gedrag weten, de details zijn onbelangrijk. Een voorbeeld in de natuurkunde is bijvoorbeeld het elektrisch geleidingsvermogen van de koperdraad, een *macroscopische* eigenschap die het resultaat is van de *microscopische* interacties tussen al de 10^{23} elektronen. Het allerlaagste niveau, dat van de individuele elektronen, is zo goed als irrelevant.

Orde-parameter

Nu we hebben gezien dat het in veel gevallen een goed idee is ons alleen te richten op het effectieve, collectieve gedrag van veel-deeltjessystemen, hebben we een manier nodig om verschillende soorten gedrag te onderscheiden. We stellen dus bijvoorbeeld de vraag: "Wat maakt een vloeistof een vloeistof en niet een gas?" Een vloeistof heeft net zo min als een gas een regelmatige structuur, zoals een kristal dat wel heeft, dus dat valt af. Daarentegen heeft een vloeistof een voorkeursdichtheid: een vloeistof zal samendrukking tegenwerken, terwijl een gas dat niet of nauwelijks doet. De onsamendrukbaarheid is dus een goede grootheid om een vloeistof van een gas te onderscheiden. Omdat de onsamendrukbaarheid van een vloeistof zeer groot is, en van een gas veel kleiner, noemen we de vloeistof *meer geordend* dan een gas.

De onsamendrukbaarheid heet in deze context de *orde-parameter*. Die laatste is een grootheid of functie die nul is voor de ongeordende toestand, en niet nul oftewel *eindig* voor de geordende toestand. Een groot deel van het vakgebied dat tegenwoordig *gecondenseerde materie* heet, gaat over het precies definiëren en meten van orde-parameters. Het ontstaan van een voorkeurswaarde van de orde-parameter en daarmee de geordende toestand heet een *fase-overgang*, bijvoorbeeld het condenseren van waterdamp (gas) tot water (vloeistof).

Dit proefschrift gaat in essentie over faseovergangen. Normaal gesproken wordt er gekeken naar de overgang van minder naar meer orde, zoals in het bovenstaande voorbeeld. Maar het is ook mogelijk om te starten vanuit een geordende toestand, bijvoorbeeld ijs, en te zien hoe dat smelt tot in dit geval water. Hierbij gaan we dus van meer naar minder orde. Een



Figuur 7.1: (a) Een regelmatig metaalrooster (b) Rooster met op twee plaatsen 'dislocaties' (rood) die de regelmatigheid verstoren. (c) Het volledig verloren gaan van het rooster, oftewel smelten, is duaal gezien de opeenhoping van dislocaties.

ander voorbeeld is hoe een ijzermagneet boven 770°C 'smelt' tot een niet-magnetische toestand. Dit is een *duale* manier van kijken naar de faseovergang: in plaats van hoe individuele deeltjes zich samen ordenen, concentreer je je op het verstoren van de totale geordende toestand. Het uitgangspunt is precies omgekeerd, maar de uitkomsten zijn volledig equivalent en net zo geldig. In sommige gevallen blijkt de duale manier, dus het ontstaan van wanorde door verstoringen in tegenstelling tot het ordenen van deeltjes, veel krachtiger dan andersom.

Vortices

Neem een paperclip in twee handen, en buig die een keer of tien. Hij zal breken op het buigpunt. Dit komt doordat het regelmatige metaalrooster bij iedere buiging verstoord wordt, zie figuur 7.1(b). Verantwoordelijk voor die verstoringen zijn zogenaamde 'dislocaties', de eindpunten van halve roosterlijnen die midden in het materiaal ophouden. Deze kosten veel energie om te maken, door de kracht die je met je handen uitoefent, maar kunnen daardoor ook niet zomaar verdwijnen. De *metaalmoeheid* is dus de verzameling van dislocaties die altijd in het materiaal aanwezig blijven. Bij het breken van de paperclip gaan er teveel roosterverbindingen verloren om hem bijeen te houden, zie figuur 7.1(c). Duaal gezien ontstaan er juist heel veel dislocaties: de teloorgang van de roosterverbindingen is dus als de opeenhoping van dislocaties.

Een dislocatie is een zogenaamd *topologisch defect*, waarbij topologisch betekent dat de effecten ervan door het hele materiaal merkbaar zijn. Een ander elementair topologisch defect is een *vortex*, zie figuur 2.1 op pagina

Figuur 7.2: Twee vortices in de vorm van een waterhoos boven de Waddenzee. Dit zijn geen puntdeeltjes maar dynamische lijnobjecten.

21. Een kolk in je badkuip en een wervelstorm zijn ook vortices (zie figuur 7.2). De kolk bestaat alleen in de vorm van het resulterende collectieve gedrag van een heleboel watermoleculen, en is dus inherent *niet-locaal*. Dat betekent dat je de snelheden van watermoleculen op een heleboel plekken moet weten voordat je kunt concluderen dat er een vortex bestaat. Toch is het wiskundig mogelijk om het middelpunt van de kolk als een locaal object te beschouwen, net zoals we dat voor de dislocaties in figuur 7.1 ook hadden kunnen doen.

Supergeleiding

Zoals opgemerkt op pagina iii, is het precies 100 jaar geleden dat supergeleiding in Leiden werd ontdekt. Vrijwel alle elementaire metalen worden bij zeer lage temperatuur supergeleidend, wat betekent dat zij i) alle elektrische weerstand verliezen, en daardoor eeuwigdurende stromen kunnen herbergen, en ii) alle magneetvelden uit hun binnenste verdrijven. In sommige supergeleiders, die met de ongeïnspireerde naam *type-II* worden aangeduid, kan een magneetveld juist wel, in de vorm van vortexlijnen, het materiaal binnendringen. Of een supergeleider type-II is, hangt af van het materiaal, en is eigenlijk niet vooraf te voorspellen.

Supergeleiding is een gevolg van de collectieve wisselwerking tussen de elektronen en het metaalrooster, en daardoor heel goed met een orde-parameter te beschrijven, die in feite aan ieder punt in het materiaal een (virtuele) richting toewijst, oftewel een pijltje. In de supergeleidende toestand wijzen deze pijlen alle dezelfde kant op, het is dus een geordende toestand. Verstoringen in die orde-parameter kunnen in de vorm van vortices optreden, zoals op de omslag is uitgebeeld. Dit kan onder invloed van een magneetveld, maar kan ook spontaan gebeuren. De fase-overgang van supergeleidend naar 'normaal' kunnen we duaal dus opvatten als de opeenhoping van vortices. In termen van die vortices belanden we dan juist in een (duaal) geordende toestand, die we vortexvloeistof noemen, omdat die ook een eindige onsamendrukbaarheid heeft.

Dit proefschrift

In dit proefschrift wordt de duale beschrijving van fase-overgangen, dus het verstoren van ordening door de opeenhoping van vortices, volledig omarmd en vervolgens uitgebreid.

Hoofdstuk 3 omschrijft de meest fundamentele vooruitgang: het generaliseren van deze vortex-dualiteit van twee naar drie ruimtedimensies plus tijd als vierde dimensie (dit heet "3+1" dimensies). In het platte vlak, dus in twee dimensies, is een vortex of dislocatie als een puntdeeltje. Maar net als de wervelstorm is een vortex in drie dimensies als een lijnvormig object. Je kunt je wel voorstellen dat een beschrijving van puntdeeltjes veel eenvoudiger is dan een beschrijving van lijnen (snaren), die zelf ook weer kunnen trillen en vervormen. Inderdaad waren de fase-overgangen in de duale, vortex-beschrijving tot op heden alleen beschikbaar in twee dimensies. Ik laat voor het eerst zien hoe dat werkt in drie dimensies. Een echte beschrijving van wat er met die vortexlijnen gebeurt is te ingewikkeld, maar we voorspellen dat het collectieve gedrag van die lijnen in feite precies hetzelfde is als het collectieve gedrag van puntdeeltjes. Omdat we dit via de dualiteit direct staven aan de gewone (niet-duale) beschrijving van een eenvoudig, veelgebruikt model, is deze voorspelling zeer plausibel.

Als je een puntdeeltje door de tijd volgt zal je een 'lijn in de geschiedenis' uittekenen, net zoals de condensatiestrepen van een vliegtuig in de lucht. Dit heet een *wereldlijn*. Maar als je een lijnobject in de tijd volgt, krijg je een tweedimensionaal *wereldoppervlak*. Deze worden in de natuurkunde van elementaire deeltjes veelvuldig gebruikt, maar in de gecondenseerde materie nauwelijks. In onze 3+1-dimensionale vortex-dualiteit zijn vortex wereldoppervlakken aan de orde van de dag. Maar omdat de kennis erover beperkt is, leek het een goed idee deze eerst toe te passen op het bekende probleem van de magnetische vortices in supergeleiders. Dit levert een fraaie en compacte wiskundige beschrijving ervan op, die alle bekende effecten gerelateerd aan zulke vortices in één enkele vergelijking kan vatten (vergelijking (4.43) in hoofdstuk 4). Deze wereldoppervlakbenadering is zo aantrekkelijk dat zij zelfs in het onderwijs over vortices in supergeleiders gebruikt zou kunnen worden.

De algemene generalisatie van vortex-dualiteit in hogere dimensies van hoofdstuk 3 wordt in hoofdstuk 5 toegepast op het specifieke geval van supergeleiders. De supergeleidende toestand is geordend, en die orde wordt verstoord door de toename van vortices. Uiteindelijk zullen we door de faseovergang in een elektrisch isolerende toestand genaamd *Mott isolator* terecht komen. Daarin hebben de ladingsdragers onderling een sterke afstotende wisselwerking en komen daardoor 'vast' te zitten, zodat ze niet meer vrij kunnen bewegen en lading transporteren. Volgens de vortex-dualiteit heeft deze isolerende toestand weer zijn eigen vortices. Deze zijn gequantiseerde lijnen van elektrische stroom. Dus net zoals een supergeleider magneetveld verdrijft, maar er vortexlijnen van magneetveld doorheen kunnen gaan, verdrijft een Mott isolator elektrische stroom maar blijkt wel stroomdraadjes te vormen onder invloed van een van buitenaf opgelegde stroom. Het woord 'gequantiseerd' betekent dat de hoeveelheid stroom door ieder draadje niet variabel is, maar een vaste waarde heeft. De totale stroom kan dus alleen toenemen door meer vortexlijnen te maken, niet door de stroom per lijn te verhogen. Ook kan de stroom pas gaan lopen als de eerste vortexlijn gevormd wordt, dus boven een bepaalde drempelwaarde voor de stroom. Dit is een vrij harde voorspelling, en in figuur 5.4 op pagina 109 wordt een aantal experimenten gesuggereerd dat dit nieuwe, verrassende fenomeen zou moeten kunnen bevestigen. Het mooiste wat je als theoreticus kan bereiken, is het voorspellen van een nieuw natuurverschijnsel dat vervolgens experimenteel bevestigd wordt.

Het laatste hoofdstuk 6 is meer inzichtelijk dan voorspellend van toon. Enkele achterliggende principes die zich in de vortex-dualiteit steeds opdringen, worden toegelicht en met elkaar verbonden. Deze principes zijn wiskundig van aard—en daardoor wat lastiger in een paar regels uit te leggen—maar onthullen belangrijke structuren die ten grondslag liggen aan het gedrag van natuur zoals wij die ervaren. Het gaat in feite over *behouden grootheden*, maar vooral over hoe die zich aan weerszijden van de vortex-dualiteit manifesteren. Gebruikmakend van deze algemene principes kunnen we een eerste uitbreiding voorzien: die naar quantum kristallen en hoe die door opeenhoping van dislocaties smelten tot quantum vloeibare kristallen. De vortex-dualiteit is zo ruimer toepasbaar dan alleen op supergeleiders, en lijkt te moeten gelden voor de meeste orde–wanorde fase-overgangen.

List of Publications

Condensing Nielsen–Olesen strings and the vortex–boson duality in 3+1 and higher dimensions
A.J. Beekman, D. Sadri and J. Zaanen
New Journal of Physics **13** 033004 (2011) — arXiv:1006.2267

Electrodynamics of Abrikosov vortices: the Field Theoretical Formulation
A.J. Beekman and J. Zaanen
accepted for Frontiers of Physics — arXiv:1106.3946

The emergence of gauge invariance: the stay-at-home gauge versus local–global duality
J. Zaanen and A.J. Beekman
submitted to Annals of Physics — arXiv:1108.2791

Type-II Mott insulators
A.J. Beekman and J. Zaanen
in preparation

Curriculum vitæ

On the 21st of November, 1979, I was born in Gouda, the Netherlands, where I completed secondary education at the Coornhert Gymnasium in 1997. The following year I spent working and travelling in France and Israel.

I entered the β/γ-propædeuse at the University of Amsterdam, through which program I proceeded to study physics. I graduated as Master of Science in theoretical physics on the thesis *Quantum double symmetries of the even dihedral groups and their breaking* under the supervision of prof. dr. ir. F.A. Bais in 2005. During this time I competed in race rowing, and also organized the Dutch Indoor Rowing Championships for five years.

In 2006 I started PhD research under the supervision of prof. dr. J. Zaanen at the Instituut-Lorentz for theoretical physics, which is part of the Leiden Institute of Physics at Leiden University. During this time I was a teaching assistant for the courses *Advanced Theory of Condensed Matter* by prof. dr. J. Zaanen, *Field Theory* by prof. dr. P.J. van Baal and *Theory of Condensed Matter* by dr. D.I. Santiago. From 2007 to 2011 I was a member of the PhD council of the Dutch Research School of Theoretical Physics.

Since 2007 I am a board member of the Amsterdam Rowing Association. In 2009 I was awarded the Membership of Merit from the Amsterdam Student Rowing Club "Nereus".

Acknowledgments

I wish to thank Ana Achúcarro, Dr Lev Bulaevskii, Vladimir Cvetković, Igor Herbut, Vladimir Juricic, Peter Kes, Yan Liu, Andrej Mesaros, Louk Rademaker, Dr Soo-Jong Rey, Jan van Ruitenbeek, Darius Sadri, Koenraad Schalm, Jian-Huang She, Ya-Wen Sun, Zlatko Tešanović, Kip Thorne and Jasper van Wezel for enlightening discussions.

I would like extend special thanks to Rolf de Jonker for the cover design, and to the proprietors of La Grande Journée in Verbier, Switzerland, where the majority of this work was produced, for their hospitality. Thanks to Tako Bruinsma and Berghauser Pont Publishing for managing the printing of this thesis.

I have been enjoying my time at the Instituut-Lorentz voor theoretische natuurkunde so much that I have extended my residence for quite a while. I thank all my colleagues in past and present, and the support staff Marianne, Fran and Trudy for an enormously stimulating work environment.

The Dutch Research School of Theoretical Physics is a prime example of an organization that is exactly suited for the geographical scale and topical scope it represents, and is instrumental in keeping a wider view within an ever more specializing and divided scientific world. I am glad to have been able to contribute to its progress.

Finally, I thank my parents, sister and brother for being there.